R. Walker

THE ELEMENTS OF EUCLID,

VIZ.

THE FIRST SIX BOOKS,

TOGETHER WITH THE

ELEVENTH AND TWELFTH.

The Errors, by which THEON, or others, have long ago Vitiated these Books, are Corrected,

And some of EUCLID's Demonstrations are Restored.

ALSO,

THE BOOK OF EUCLID'S DATA,

In like manner Corrected.

BY

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Emeritus Professor of Mathematics in the University of Glasgow.

To this FOURTH EDITION are also annexed,

ELEMENTS of PLAIN and SPHERICAL TRIGONOMETRY.

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BRITISH MUSEUM

TO THE

KING,

THIS EDITION

OF THE

PRINCIPAL BOOKS

OF THE

ELEMENTS OF EUCLID,

AND OF THE

BOOK OF HIS DATA,

IS MOST HUMBLY DEDICATED,

BY

HIS MAJESTY'S

MOST DUTIFUL, AND

MOST DEVOTED

SUBJECT AND SERVANT,

ROBERT SIMSON.

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PREFACE.

THE Opinions of the Moderns concerning the Author of the Elements of Geometry, which go under Euclid's name, are very different and contrary to one another. Peter Ramus aſcribes the Propoſitions, as well as their Demonſtrations, to Theon; others think the Propoſitions to be Euclid's, but that the Demonſtrations are Theon's; and others maintain that all the Propoſitions and their Demonſtrations are Euclid's own. John Buteo and Sir Henry Savile are the Authors of greateſt Note who aſſert this laſt, and the greater part of Geometers have ever ſince been of this Opinion, as they thought it the moſt probable. Sir Henry Savile, after the ſeveral Arguments he brings to prove it, makes this Concluſion (Page 13. Prælect.) "That, excepting a very few "Interpolations, Explications, and Additions, Theon altered "nothing in Euclid." But, by often conſidering and comparing together the Definitions and Demonſtrations as they are in the Greek Editions we now have, I found that Theon, or whoever was the Editor of the preſent Greek Text, by adding ſome things, ſuppreſſing others, and mixing his own with Euclid's Demonſtrations, had changed more things to the worſe than is commonly ſuppoſed, and thoſe not of ſmall moment, eſpecially in the Fifth and Eleventh Books of the Elements, which this Editor has greatly vitiated; for inſtance, by ſubſtituting a ſhorter, but inſufficient Demonſtration of the 18th Prop. of the 5th Book, in place of the legittimate one which Euclid had given; and by taking out of this Book, beſides other things, the good Definition which Eudoxus or Euclid had given of Compound Ratio, and giving an abſurd one in place of it in the 5th Definition of the 6th Book, which neither Euclid, Archimedes, Appollonius, nor any Geometer before Theon's time, ever made uſe of, and of which there is not to be found the leaſt appearance in any of their Writings; and, as this Definition did much embarraſs Beginners, and is quite uſeleſs, it is now thrown out of the Elements, and another, which without doubt Euclid had given, is put in its proper place among the Definitions of the

5th

5th Book, by which the Doctrine of Compound Ratios is rendered plain and eaſy. Beſides, among the Definitions of the 11th Book, there is this, which is the 10th, viz. "Equal "and ſimilar ſolid figures are thoſe which are contained by "ſimilar planes of the ſame number and magnitude." Now, this Propoſition is a Theorem, not a Definition, becauſe the equality of figures of any kind muſt be demonſtrated, and not aſſumed; and therefore, tho' this were a true Propoſition, it ought to have been demonſtrated. But indeed this Propoſition, which makes the 10th Definition of the 11th Book, is not univerſally true, except in the caſe in which each of the ſolid angles of the figures is contained by no more than three plain angles; for, in other caſes, two ſolid figures may be contained by ſimilar planes of the ſame number and magnitude, and yet be unequal to one another; as ſhall be made evident in the Notes ſubjoined to theſe Elements. In like manner, in the Demonſtration of the 26th Prop. of the 11th Book, it is taken for granted, that thoſe ſolid angles are equal to one another which are contained by plain angles of the ſame number and magnitude, placed in the ſame order; but neither is this univerſally true, except in the caſe in which the ſolid angles are contained by no more than three plain angles; nor of this caſe is there any Demonſtration in the Elements we now have, though it be quite neceſſary there ſhould be one. Now, upon the 10th Definition of this Book depend the 25th and 28th Propoſitions of it; and upon the 25th and 26th depend other eight, viz. the 27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſame Book; and the 12th of the 12th Book depends upon the 8th of the ſame, and this 8th, and the Corollary of Propoſition 17th, and Prop. 18th of the 12th Book depend upon the 9th Definition of the 11th Book, which is not a right Definition, becauſe there may be ſolids contained by the ſame number of ſimilar plain figures, which are not ſimilar to one another, in the true ſenſe of ſimilarity received by all Geometers; and all theſe Propoſitions have, for theſe reaſons, been inſufficiently demonſtrated ſince Theon's time hitherto. Beſides, there are ſeveral other things, which have nothing of Euclid's accuracy, and which plainly ſhew that his Elements have been much corrupted by unſkilful Geometers; and, though theſe are not ſo groſs as the others now mentioned, they ought by no means to remain uncorrected.

Upon

Upon theſe Accounts it appeared neceſſary, and I hope will prove acceptable to all Lovers of accurate Reaſoning, and of Mathematical Learning, to remove ſuch Blemiſhes, and reſtore the principal Books of the Elements to their original Accuracy, as far as I was able; eſpecially ſince theſe Elements are the Foundation of a Science by which the Inveſtigation and Diſcovery of uſeful Truths, at leaſt in Mathematical Learning, is promoted as far as the limited Powers of the Mind allow; and which likewiſe is of the greateſt Uſe in the Arts both of Peace and War, to many of which Geometry is abſolutely neceſſary. This I have endeavoured to do by taking away the inaccurate and falſe Reaſonings which unſkilful Editors have put into the place of ſome of the genuine Demonſtrations of Euclid, who has ever been juſtly celebrated as the moſt accurate of Geometers, and by reſtoring to him thoſe Things which Theon or others have ſuppreſſed, and which have theſe many Ages been buried in Oblivion.

In this third Edition, Ptolemy's Propoſition concerning a Property of quadrilateral figures in a circle is added at the end of the ſixth Book. Alſo the Note on the 29th Prop. Book 1ſt, is altered, and made more explicit, and a more general Demonſtration is given inſtead of that which was in the Note on the 10th Definition of Book 11th; beſides the Tranſlation is much amended by the friendly aſſiſtance of a learned Gentleman.

To which is alſo added, the Elements of Plain and Spherical Trigonometry, which are commonly taught after the Elements of Euclid.

THE ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I.

A Point is that which hath no parts, or which hath no magnitude. See Notes.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A ſtraight line is that which lies evenly between its extreme points.

V.

A ſuperficies is that which hath only length and breadth.

VI.

The extremities of a ſuperficies are lines.

VII.

A plane ſuperficies is that in which any two points being taken, the ſtraight line between them lies wholly in that ſuperficies. See N.

VIII.

"A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the "ſame direction." See N.

IX.

A plane rectilineal angle is the inclination of two ſtraight lines to one another, which meet together, but are not in the ſame ſtraight line.

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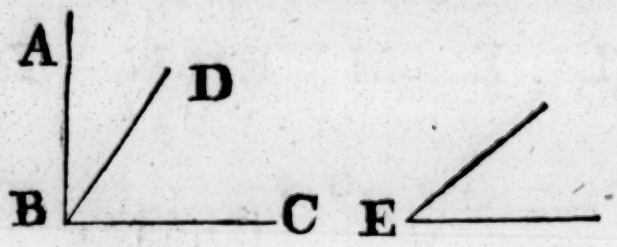
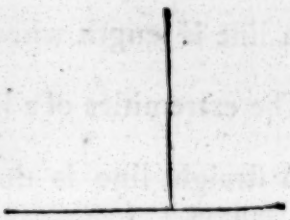
Book I.



'N. B. When ſeveral angles are at one point B, any one 'of them is expreſſed by three letters, of which the letter that 'is at the vertex of the angle, that is, at the point in which 'the ſtraight lines that contain the angle meet one another, is 'put between the other two letters, and one of theſe two is 'ſomewhere upon one of thoſe ſtraight lines, and the other 'upon the other line: Thus the angle which is contained by 'the ſtraight lines AB, CB is named the angle ABC, or CBA; 'that which is contained by AB, DB is named the angle 'ABD, or DBA; and that which is contained by DB, CB is 'called the angle DBC, or CBD; but if there be only one angle 'at a point, it may be expreſſed by a letter placed at that point; 'as the angle at E.'

X

When a ſtraight line ſtanding on another ſtraight line makes the adjacent angles equal to ane another, each of the angles is called a right angle; and the ſtraight line which ſtands on the other is called a perpendicular to it.



XI.

An obtuſe angle is that which is greater than a right angle.

XII.

An acute angle is that which is leſs than a right angle.

XIII.

"A term or boundary, is the extremity of any thing."

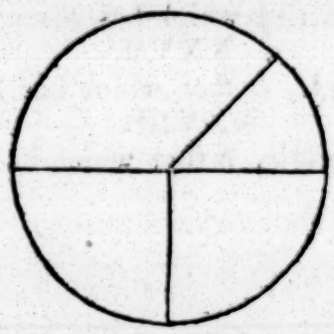
XIV.

A figure is that which is incloſed by one or more boundaries.

A circle

XV.

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another:



XVI.

And this point is called the center of the circle.

XVII.

A diameter of a circle is a straight line drawn thro' the center, and terminated both ways by the circumference. See N.

XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

"A segment of a circle is the figure contained by a straight "line and the circumference it cuts off."

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines:

XXII.

Quadrilateral, by four straight lines:

XXIII.

Multilateral figures, or Polygons, by more than four straight lines.

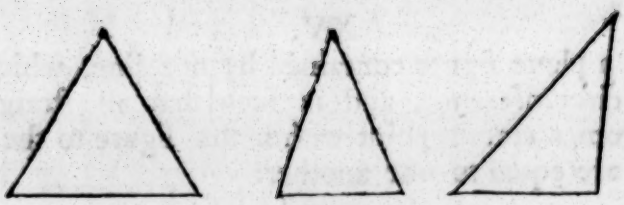
XXIV.

Of three sided figures, an equilateral triangle is that which has three equal sides.

XXV.

An isosceles triangle, is that which has only two sides equal.

XXVI.



XXVI.

A ſcalene triangle, is that which has three unequal ſides.

XXVII.

A right angled triangle, is that which has a right angle.

XXVIII.

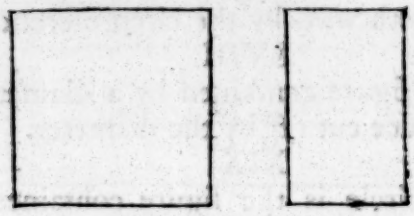
An obtuſe angled triangle, is that which has an obtuſe angle.

XXIX.

An acute angled triangle, is that which has three acute angles.

XXX.

Of four ſided figures, a ſquare is that which has all its ſides equal, and all its angles right angles.

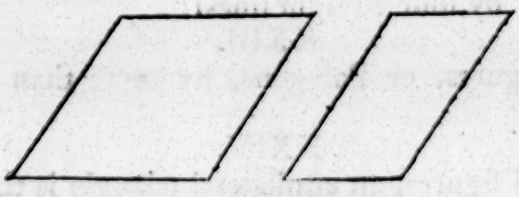


XXXI.

An oblong is that which has all its angles right angles, but has not all its ſides equal.

XXXII.

A rhombus is that which has all its ſides equal, but its angles are not right angles.



XXXIII.

See N. A rhomboid is that which has its oppoſite ſides equal to one another, but all its ſides are not equal, nor its angles right angles.

XXXIV.

XXXIV.

All other four ſided figures beſides theſe, are called Trapeziums.

XXXV.

Parallel ſtraight lines, are ſuch as are in the ſame plane, and which, being produced ever ſo far both ways, do not meet.

POSTULATES.

I.

LET it be granted that a ſtraight line may be drawn from any one point to any other point.

II.

That a terminated ſtraight line may be produced to any length in a ſtraight line.

III.

And that a circle may be deſcribed from any center, at any diſtance from that center.

AXIOMS.

I.

THINGS which are equal to the ſame are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the ſame, are equal to one another.

VII.

Things which are halves of the ſame, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the ſame ſpace, are equal to one another.

Book I.

IX.

The whole is greater than its part.

X.

Two ſtraight lines cannot incloſe a ſpace.

XI.

All right angles are equal to one another.

XII.

"If a ſtraight line meets two ſtraight lines, ſo as to make the "two interior angles on the ſame ſide of it taken together "leſs than two right angles, theſe ſtraight lines being con- "tinually produced, ſhall at length meet upon that ſide on "which are the angles which are leſs than two right angles. "See the notes on Prop. 29. of Book I."

PROPO-

PROPOSITION I. PROBLEM.

Book 1.

TO describe an equilateral triangle upon a given finite straight line.

Let AB be the given straight line; it is required to describe an equilateral triangle upon it.

From the center A, at the distance AB, describe[a] the circle BCD, and from the center B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines[b] CA, CB to the points A, B; ABC shall be an equilateral triangle.

a. 3d Postulate.

b. 1st Post.

Because the point A is the center of the circle BCD, AC is equal[c] to AB; and because the point B is the center of the circle ACE, BC is equal to BA: But it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB; but things which are equal to the same are equal to one another[d]; therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB. Which was required to be done.

c. 15th Definition.

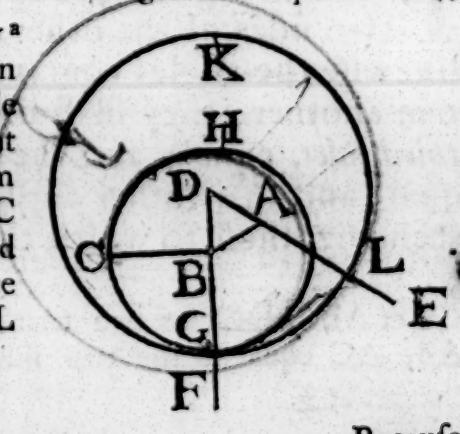
d. 1st Axiom.

PROP. II. PROB.

FROM a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line; it is required to draw from the point A a straight line equal to BC.

From the point A to B draw[a] the straight line AB; and upon it describe[b] the equilateral triangle DAB, and produce[c] the straight lines DA, DB to E and F; from the center B, at the distance BC describe[d] the circle CGH, and from the center D, at the distance DG describe the circle GKL. AL shall be equal to BC.



a. 1. Post.

b. 1. 1.

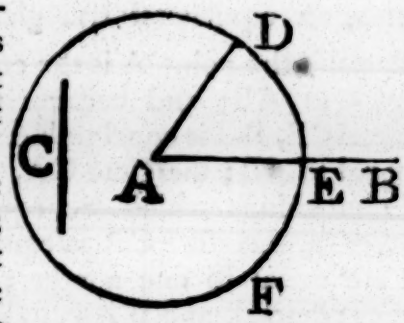
c. 2. Post.

Post.

Book I. Becauſe the point B is the center of the circle CGH, BC is equal [e] to BG; and becauſe D is the center of the circle GKL,
e 5 Def. DL is equal to DG, and DA, DB, parts of them, are equal;
f. 3, Ax therefore the remainder AL is equal to the remainder [f] BG: But it has been ſhewn, that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the ſame are equal to one another; therefore the ſtraight line AL is equal to BC. Wherefore from the given point A a ſtraight line AL has been drawn equal to the given ſtraight line BC. Which was to be done.

PROP. III. PROB.

FROM the greater of two given ſtraight lines to cut off a part equal to the leſs.



Let AB and C be the two given ſtraight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C the leſs.

a. 2. 1. From the point A draw [a] the ſtraight line AD equal to C; and from the center A, and at
b. 3. Poſt. the diſtance AD, deſcribe [b] the circle DEF; and becauſe A is the center of the circle DEF, AE ſhall be equal to AD; but the ſtraight line C is likewiſe equal to AD; whence AE and C are each of them equal to AD; wherefore the ſtraight line AE is equal
c. 1. Ax. to [c] C, and from AB, the greater of two ſtraight lines, a part AE has been cut off equal to C the leſs. Which was to be done.

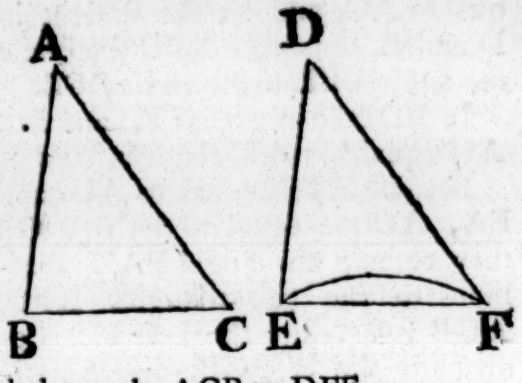
PROP. IV. THEOREM.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each; and have likewiſe the angles contained by thoſe ſides equal to one another; they ſhall likewiſe have their baſes, or *third ſides*, equal; and the two triangles ſhall be equal; and their other angles ſhall be equal, each to each, viz. thoſe to which the equal ſides are oppoſite.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz.

AB

AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, the baſe BC ſhall be equal to the baſe EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal ſides are oppoſite, ſhall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



For, if the triangle ABC be applied to DEF ſo that the point A may be on D, and the ſtraight line AB upon DE; the point B ſhall coincide with the point E, becauſe AB is equal to DE; and AB coinciding with DE, AC ſhall coincide with DF, becauſe the angle BAC is equal to the angle EDF; wherefore alſo the point C ſhall coincide with the point F, becauſe the ſtraight line AC is equal to DF: But the point B coincides with the point E; wherefore the baſe BC ſhall coincide with the baſe EF, becauſe the point B coinciding with E, and C with F, if the baſe BC does not coincide with the baſe EF, two ſtraight lines would incloſe a ſpace, which is impoſſible. [a] Therefore the baſe BC ſhall coincide with the baſe EF, and be equal to it. Wherefore the whole triangle ABC ſhall coincide with the whole triangle DEF, and be equal to it; and the other angles of the one ſhall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore, if two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe the angles contained by thoſe ſides equal to one another, their baſes ſhall likewiſe be equal, and the triangles be equal, and their other angles to which the equal ſides are oppoſite ſhall be equal, each to each. Which was to be demonſtrated.

a. 10. Ax

PROP. V. THEOR.

THE angles at the baſe of an Iſoſceles triangle are equal to one another; and if the equal ſides be produced, the angles upon the other ſide of the baſe ſhall be equal.

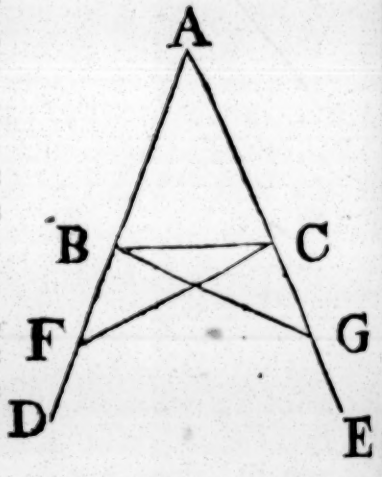
Let ABC be an Iſoſceles triangle, of which the ſide AB is equal

Book I.

qual to AC, and let the ſtraight lines AB, AC be produced to D and E, the angle ABC ſhall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE, the greater, cut off
a. 3. 1. AG equal [a] to AF, the leſs, and join FC, BG.

Becauſe AF is equal to AG, and AB to AC; the two ſides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the baſe FC is e-
b. 4. 1. qual [b] to the baſe GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal [b] to the remaining angles of the other, each to each, to which the equal ſides are oppoſite; viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And becauſe the whole AF is equal to the whole AG, of which the parts AB, AC are equal; the re-



c. 3. Ax. mainder BF ſhall be equal [c] to the remainder CG; and FG was proved to be equal to GB; therefore the two ſides BF, FC are equal to the two CG, GB, each to each; and the angle BFC is equal to the angle CGB; wherefore the triangles are equal [b], and their remaining angles, each to each, to which the equal ſides are oppoſite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG: And, ſince it has been demonſtrated that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF, are alſo equal; the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the baſe of the triangle ABC: And it has alſo been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other ſide of the baſe. Therefore the angles at the baſe, &c. Q. E. D.

COROLLARY. Hence every equilateral triangle is alſo equiangular.

PROP. VI. THEOR.

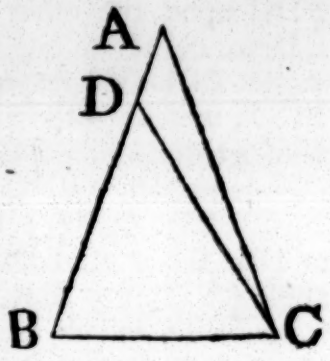
IF two angles of a triangle be equal to one another, the ſides alſo which ſubtend, *or are oppoſite to*, the equal angles ſhall be equal to one another.

Let

Let ABC be a triangle having the angle ABC equal to the angle ACB; the ſide AB is alſo equal to the ſide AC. Book I.

For if AB be not equal to AC, one of them is greater than the other: Let AB be the greater, and from it cut [a] off, DB equal to AC, the leſs, and join DC; therefore, becauſe in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two ſides DB, BC are equal to the two AC, CB, each to each; and the angle DBC is equal to the angle ACB; therefore the baſe DC is equal to the baſe AB, and the triangle DBC is equal to the triangle [b] ACB, the leſs to the greater; which is abſurd. Therefore AB is not unequal to AB, that is, it is equal to it. Wherefore, if two angles, &c. Q. E. D.

a. 3. 1.



b. 4. 1.

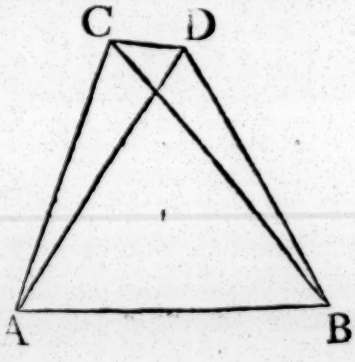
Cor. Hence every equiangular triangle is alſo equilateral.

PROP. VII. THEOR.

UPon the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. See N.

If it be poſſible, let there be two triangles ACB, ADB upon the ſame baſe AB, and upon the ſame ſide of it, which have their ſides CA, DA, terminated in the extremity A of the baſe, equal to one another, and likewiſe their ſides CB, DB that are terminated in B.

Join CD; then, in the caſe in which the vertex of each of the triangles is without the other triangle, becauſe AC is equal to AD, the angle ACD is equal [a] to the angle ADC; But the angle ACD is greater than the angle BCD; therefore the angle ADC is greater alſo than BCD; much more then is the angle BDC greater than the angle BCD. Again, becauſe CB is equal to DB, the angle BDC is equal [a] to the angle BCD; but it has been demonſtrated to be greater than it; which is impoſſible.



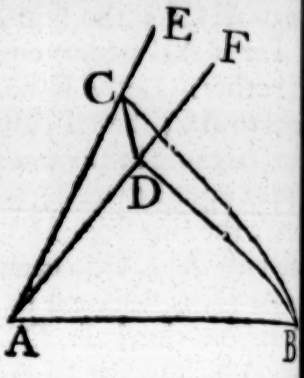
a. 5. 1.

But

Book I.

But if one of the vertices, as D, be within the other triangle ACB; produce AC, AD to E, F; therefore, becauſe AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other ſide of the baſe CD are equal [a] to one another; but the angle ECD is greater than the angle BCD; wherefore the angle FDC is likewiſe greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, becauſe CB is equal to DB, the angle BDC is equal [a] to the angle BCD; but BDC has been proved to be greater than the ſame BCD, which is impoſſible. The caſe in which the vertex of one triangle is upon a ſide of the other needs no demonſtration.

a. 5. 1.

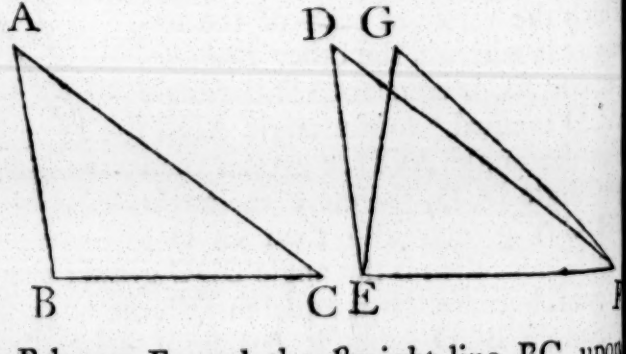


Therefore upon the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. Q. E. D.

PROP. VIII. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe their baſes equal; the angle which is contained by the two ſides of the one ſhall be equal to the angle contained by the two ſides equal to them of the other.

Let ABC, DEF be two triangles having the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and alſo the baſe BC equal to the baſe EF. The angle BAC is equal to the angle EDF.



For, if the triangle ABC be applied to DEF, ſo that the point B be on E, and the ſtraight line BC upon EF; the point C ſhall alſo coincide with the point F, becauſe BC

C is equal to EF; therefore BC coinciding with EF, BA and AC ſhall coincide with ED and DF; for if the baſe BC coincides with the baſe EF, but the ſides BA, CA do not coincide with the ſides ED, FD, but have a different ſituation, as EG, FG; then, upon the ſame baſe EF, and upon the ſame ſide of it, there can be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe their ſides terminated in the other extremity: But this is impoſſible [a]; therefore, if the baſe BC coincides with the baſe EF, the ſides BA, AC cannot but coincide with the ſides ED, DF; wherefore likewiſe the angle BAC coincides with the angle EDF, and is equal [b] to it. Therefore if two triangles, &c. Q. E. D. Book I.

a. 7. 1.

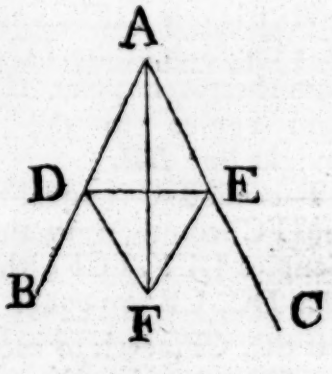
b. 8. Ax.

PROP. IX. PROB.

TO biſect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to biſect it.

Take any point D in AB, and from AC cut [a] off AE equal to AD; join DE, and upon it deſcribe [b] an equilateral triangle DEF; then join AF, the ſtraight line AF biſects the angle BAC.



Becauſe AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two ſides DA, AF are equal to the two ſides EA, AF, each to each; and the baſe DF is equal to the baſe EF; therefore the angle DAF is equal [c] to the angle EAF; wherefore the given rectilineal angle BAC is biſected by the ſtraight line AF. Which was to be done.

a. 3. 1.

b. 1. 1.

c. 8. 1.

PROP. X. PROB.

TO biſect a given finite ſtraight line, that is, to divide it into two equal parts.

Let AB be the given ſtraight line; it is required to divide it into two equal parts.

Deſcribe [a] upon it an equilateral triangle ABC, and biſect the angle ACB by the ſtraight line CD. AB is cut into two equal parts in the point D.

a. 1. 1.

b. 9. 1.

Becauſe

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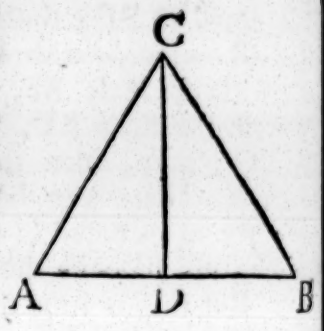
Becauſe AC is equal to CB, and CD common to the two triangles ACD, BCD; the two ſides AC, CD are equal to BC, CD, each to each; and the angle ACD is equal to the angle BCD; therefore the baſe AD is equal to the baſe [c] DB, and the ſtraight line AB is divided into two equal parts in the point D. Which was to be done.

c. 4. 1.



PROP. XI. PROB.

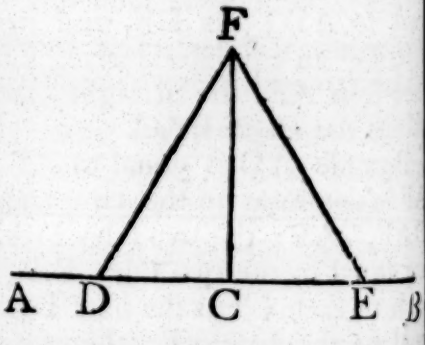
TO draw a ſtraight line at right angles to a given ſtraight line, from a given point in the ſame.

See N.

Let AB be a given ſtraight line, and C a point given in it; it is required to draw a ſtraight line from the point C at right angles to AB.

a. 3. 1.
b. 1. 1.

Take any point D in AC, and make [a] CE equal to CD, and upon DE deſcribe [b] the equilateral triangle DFE, and join FC; the ſtraight line FC drawn from the given point C is at right angles to the given ſtraight line AB.



Becauſe DC is equal to CE, and FC common to the two triangles DCF, ECF; the two ſides DC, CF are equal to the two EC, CF, each to each; and the baſe DF is equal to the baſe EF; therefore the angle DCF is equal [c] to the angle ECF; and they are adjacent angles. But, when the adjacent angles which one ſtraight line makes with another ſtraight line are equal to one another, each of them is called a right [d] angle; therefore each of the angles DCF, ECF is a right angle. Wherefore from the given point C in the given ſtraight line AB, FC has been drawn at right angles to AB. Which was to be done.

c. 8. 1.

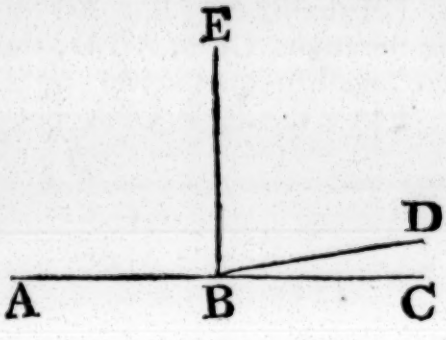
d. 10. Def. 1.

Cor. By help of this Problem it may be demonſtrated that two ſtraight lines cannot have a common ſegment.

If it be poſſible, let the two ſtraight lines ABC, ABD have the ſegment AB common to both of them. From the point B draw BE at right angles to AB; and becauſe ABC is a ſtraight line,

Book I.

ine, the angle CBE is equal[a] to he angle EBA; in the ſame manner, becauſe ABD is a traight line, the angle DBE is qual to the angle EBA; where- ore the angle DBE is equal to he angle CBE, the leſs to the reater; which is impoſſible; herefore two ſtraight lines can- ot have a common ſegment.

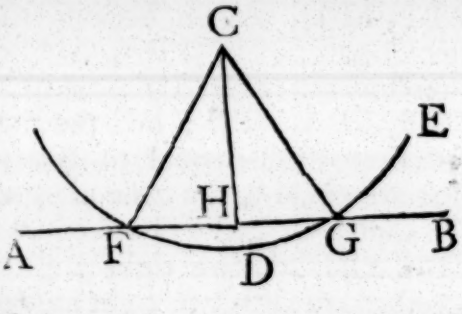


a. 10. Def. 1.

PROP. XII. PROB.

TO draw a ſtraight line perpendicular to a given ſtraight line of an unlimited length, from a given point without it.

Let AB be the given ſtraight line, which may be produced o any length both ways, and let C be a point without it. It is equired to draw a ſtraight line erpendicular to AB from the oint C.



Take any point D upon the ther ſide of AB, and from he center C, at the diſtance CD, deſcribe[b] the circle EGF meeting AB in F, G; and bi- ect[c] FG in H, and join CF, CH, CG, the ſtraight line CH drawn from the given point C, s perpendicular to the given ſtraight line AB.

b. 3. Poſt.

c. 10. 1.

Becauſe FH is equal to HG, and HC common to the two riangles FHC, GHC, the two ſides FH, HC are equal to the wo GH, HC, each to each; and the baſe CF is equal[d] to the baſe CG; therefore the angle CHF is equal[e] to the angle CHG; nd they are adjacent angles; but when a ſtraight line ſtanding n a ſtraight line makes the adjacent angles equal to one ano- her, each of them is a right angle, and the ſtraight line which tands upon the other is called a perpendicular to it; therefore rom the given point C a perpendicular CH has been drawn to he given ſtraight line AB. Which was to be done.

d. 15. Def. 1.

e. 8. 1.

PROP. XIII. THEOR.

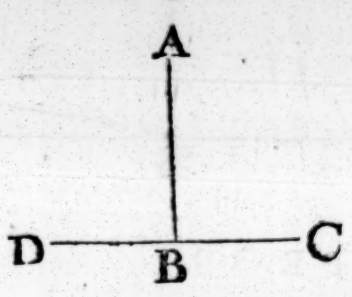
THE angles which one ſtraight line makes with another upon one ſide of it, are either two right angles, or are together equal to two right angles.

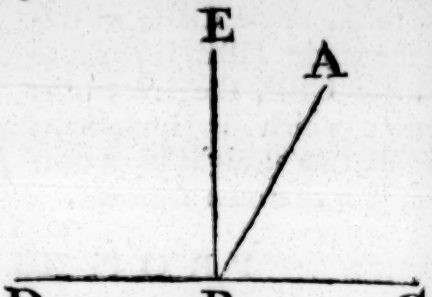
Let

Book I.

Let the ſtraight line AB make with CD, upon one ſide of it, the angles CBA, ABD; theſe are either two right angles, or are together equal to two right angles

For, if the angle CBA be equal to ABD, each of them is



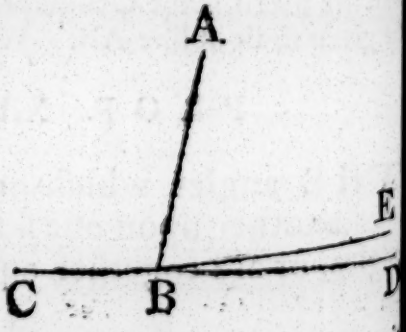


a. Def. 10. right [a] angle; but, if not, from the point B draw BE at right
b. 11. 1. angles [b] to CD; therefore the angles CBE, EBD are two right angles, and becauſe CBE is equal to the two angles CBA, ABD together; add the angle EBD to each of theſe equals; there
c. 2. Ax. fore the angles CBE, EBD are equal [c] to the three angles CBA, ABE, EBD. Again, becauſe the angle DBA is equal to the two angles DBE, EBA, add to theſe equals the angle ABC; therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC; but the angles CBE, EBD have been demonſtrated to be equal to the ſame three angles; and things
d. 1. Ax. that are equal to the ſame are equal [d] to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC; but CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles. Wherefore when a ſtraight line, &c. Q. E. D.

PROP. XIV. THEOR.

IF at a point in a ſtraight line, two other ſtraight lines upon the oppoſite ſides of it, make the adjacent angles together equal to two right angles, theſe two ſtraight lines ſhall be in one and the ſame ſtraight line.

At the point B in the ſtraight line AB, let the two ſtraight lines BC, BD upon the oppoſite ſides of AB, make the adjacent angles ABC, ABD equal together to two right angles. BD is in the ſame ſtraight line with CB.



For, if BD be not in the ſame ſtraight line with CB, let BE be

in the ſame ſtraight line with it; therefore, becauſe the ſtraight line AB makes angles with the ſtraight line CBE, upon one ſide of it, the angles ABC, ABE are together equal [a] to two right angles; but the angles ABC, ABD are likewiſe together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA ABD: Take away the common angle ABC, the remaining angle ABE is equal [b] to the remaining angle ABD, the leſs to the greater, which is impoſſible; therefore BE is not in the ſame ſtraight line with BC. And, in like manner, it may be demonſtrated, that no other can be in the ſame ſtraight line with it but BD; which therefore is in the ſame ſtraight line with CB. Wherefore if at a point, &c. Q. E. D.

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a. 13. 1.

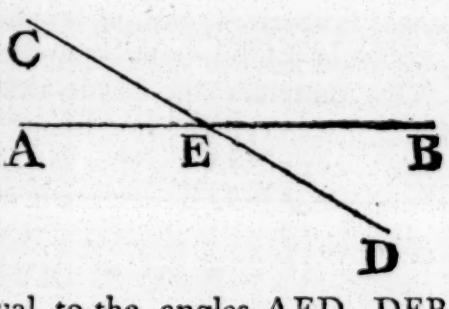
b. 3. Ax.

PROP. XV. THEOR.

IF two ſtraight lines cut one another, the vertical, *or oppoſite*, angles ſhall be equal.

Let the two ſtraight lines AB, CD cut one another in the point E; the angle AEC ſhall be equal to the angle DEB, and CEB to AED.

Becauſe the ſtraight line AE makes with CD the angles CEA, AED, theſe angles are together equal [a] to two right angles. Again, becauſe the ſtraight line DE makes with AB the angles AED, DEB; theſe alſo are together equal [a] to two right angles; and CEA, AED have been demonſtrated to be equal to two right angles; wherefore



a. 13. 1.

the angles CEA, AED are equal to the angles AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal [b] to the remaining angle DEB. In the ſame manner it can be demonſtrated that the angles CEB, AED are equal. Therefore if two ſtraight lines, &c. Q. E. D.

b. 3. Ax.

Cor. 1. From this it is manifeſt, that if two ſtraight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

Cor. 2. And conſequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

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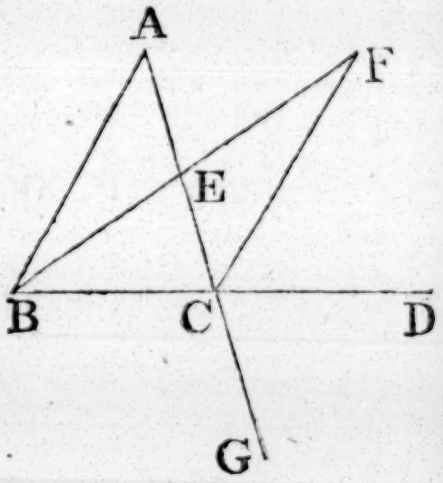
PROP. XVI. THEOR.

IF one ſide of a triangle be produced, the exterior angle is greater than either of the interior oppoſite angles.

Let ABC be a triangle, and let its ſide BC be produced to D, the exterior angle ACD is greater than either of the interior oppoſite angles CBA, BAC.



a. 10. 1. Biſect [a] AC in E, join BE and produce it to F, and make EF equal to BE; join alſo FC, and produce AC to G.

Becauſe AE is equal to EC, and BE to FF; AF, EB are equal to CE, EF, each to each; and the angle AEB is

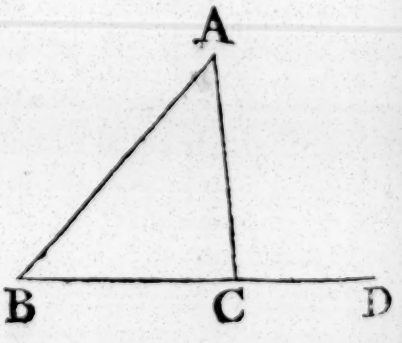
b. 15. 1. equal [b] to the angle CEF, becauſe they are oppoſite vertical angles; therefore the

c. 4. 1. baſe AB is equal [c] to the baſe CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal ſides are oppoſite; wherefore the angle BAE is equal to the angle ECF; but the angle ECD is greater than the angle ECF; therefore the angle ACD is greater than BAE: In the ſame manner, if the ſide BC be biſected, it may be demonſtrated

d. 15. 1. that the angle BCG, that is [d], the angle ACD, is greater than the angle ABC. Therefore if one ſide, &c. Q. E. D.

PROP. XVII. THEOR.

ANY two angles of a triangle are together leſs than two right angles.



Let ABC be any triangle; any two of its angles together are leſs than two right angles.

Produce BC to D; and becauſe ACD is the exterior angle of the triangle ABC, ACD is

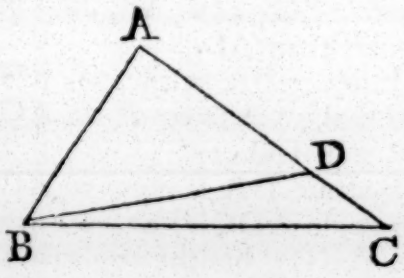
a. 16. 1. greater [a] than the interior and oppoſite angle ABC; to each of

theſe

theſe add the angle ACB, therefore the angles ACD, ACB are greater than the angles ABC, ACB; but ACD, ACB are together equal [b] to two right angles; therefore the angles ABC, BCA are leſs than two right angles In like manner it may be demonſtrated that BAC, ACB, as alſo CAB, ABC are leſs than two right angles. Therefore any two angles, &c. Q. E. D. Book. I. b. 13. 1.

PROP. XVIII. THEOR.

THE greater ſide of every triangle is oppoſite to the greater angle.



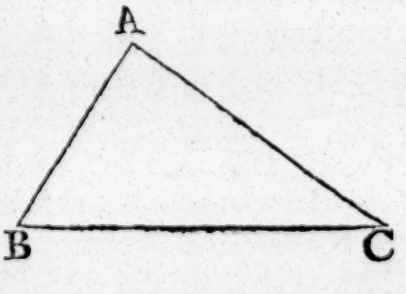
Let ABC be a triangle, of which the ſide AC is greater than the ſide AB; the angle ABC is alſo greater than the angle BCA.

Becauſe AC is greater than AB, make [a] AD equal to AB, and join BD; and becauſe ADB is the exterior angle of the triangle BDC, it is greater [b] than the interior and oppoſite angle DCB; but ADB is equal [c] to ABD, becauſe the ſide AB is equal to the ſide AD; therefore the angle ABD is likewiſe greater than the angle ACB; wherefore much more is the angle ABC greater than ACB. Therefore the greater ſide, &c. Q. E. D. a. 3. 1. b. 16. 1. c. 5. 1.

PROP. XIX. THEOR.

THE greater angle of every triangle is ſubtended by the greater ſide, or *has the greater ſide oppoſite to it.*

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA; the ſide AC is likewiſe greater than the ſide AB.



For, if it be not greater, AC muſt either be equal to AB, or leſs than it; it is not equal, becauſe then the angle ABC would be equal [a] to the angle ACB; but it is not; therefore AC is not equal to AB; neither is it leſs; becauſe then the angle a. 5. 1.

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ABC would be leſs [b] than the angle ACB; but it is not; therefore the ſide AC is not leſs than AB; and it has been ſhewn that it is not equal to AB; therefore AC is greater than AB. Wherefore the greater angle, &c. Q. E. D.

b. 18. 1.

PROP. XX. THEOR.

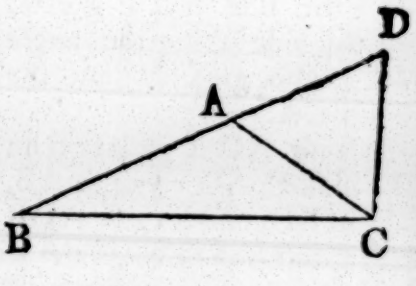
See N.

ANY two ſides of a triangle are together greater than the third ſide.

Let ABC be a triangle; any two ſides of it together are greater than the third ſide, viz. the ſides BA, AC greater than the ſide BC; and AB, BC greater than AC; and BC, CA greater than AB.

Produce BA to the point D, and make [a] AD equal to AC, and join DC.

a. 3. 1.



Becauſe DA is equal to AC, the angle ADC is likewiſe equal [b] to ACD; but the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC; and becauſe the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater [c] ſide is oppoſite to the greater angle, therefore the ſide DB is greater than the ſide BC; but DB is equal to BA and AC; therefore the ſides BA, AC are greater than BC. In the ſame manner it may be demonſtrated, that the ſides AB, BC are greater than CA; and BC, CA greater than AB. Therefore any two ſides, &c. Q. E. D.

b. 5. 1.

c. 19. 1.

PROP. XXI. THEOR.

See N.

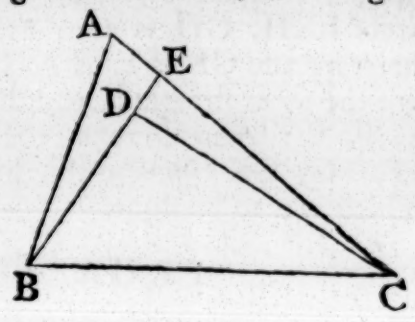
IF, from the ends of the ſide of a triangle, there be drawn two ſtraight lines to a point within the triangle, theſe ſhall be leſs than the other two ſides of the triangle, but ſhall contain a greater angle.

Let the two ſtraight lines BD, CD be drawn from B, C, the ends of the ſide BC of the triangle ABC, to the point D within it; BD and DC are leſs than the other two ſides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and becauſe two ſides of a triangle are greater than the third ſide, the two ſides BA, AE of the tri-

angle

angle ABE are greater than BE. To each of theſe add EC, Book I.
therefore the ſides BA, AC are greater than BE, EC: Again,
becauſe the two ſides CE, ED of the triangle CED are greater than CD, add DB to each of theſe; therefore the ſides CE, EB are greater than CD, DB; but it has been ſhewn that BA, AC, are greater than BE, EC; much more than are BA, AC greater than BD, DC.



Again, becauſe the exterior angle of a triangle is greater than the interior and oppoſite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the ſame reaſon, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonſtrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore if from the ends of, &c. Q. E. D.

PROP. XXII. PROB.

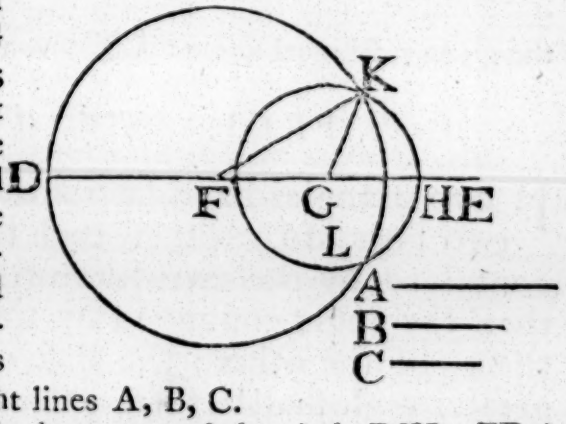
TO make a triangle of which the ſides ſhall be equal to three given ſtraight lines; but any two whatever of theſe muſt be greater than the third [a]. See N. a. 20. 1.

Let A, B, C be the three given ſtraight lines, of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle of which the ſides ſhall be equal to A, B, C, each to each.

Take a ſtraight line DE terminated at the point D, but unlimited towards E, and make [a] DF equal to A, FG to B, and GH equal to C; and from the center F, at the diſtance FD, deſcribe [b] the circle DKL; and from the center G, at the diſtance GH deſcribe [b] another circle HLK, and join KF, KG; the triangle KFG has its ſides equal to the three ſtraight lines A, B, C.

a. 3. 1.

b. 3. Poſt.



Becauſe the point F is the center of the circle DKL, FD is

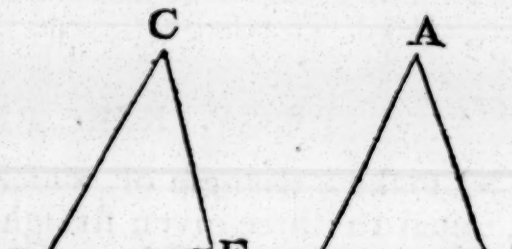
Book I.

equal [c] to FK; but FD is equal to the ſtraight line A; therefore FK is equal to A: Again, becauſe G is the center of the
c. 15. Def. circle LKH, GH is equal [c] to GK; but GH is equal to C, therefore alſo GK is equal to C; and FG is equal to B; therefore the three ſtraight lines KF, FG, GK are equal to the three A, B, C: And therefore the triangle KFG has its three ſides KF, FG, GK equal to the three given ſtraight lines A, B, C. Which was to be done.

PROP. XXIII. PROB.

AT a given point in a given ſtraight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given ſtraight line AB, that ſhall be equal to the given rectilineal angle DCE.



Take in CD, CE, any points D, E, and
a. 22. 1. join DE; and make [a] the triangle AFG the ſides of which ſhall be equal to the three ſtraight lines CD, DE, EC, ſo that CD be equal to AF, CE to AG, and DE to FG; and becauſe DC, CE are equal to FA, AG, each to each, and the baſe DE to the baſe FG; the angle DCE
b. 8. 1. is equal [b] to the angle FAG. Therefore, at the given point A in the given ſtraight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

PROP. XXIV. THEOR.

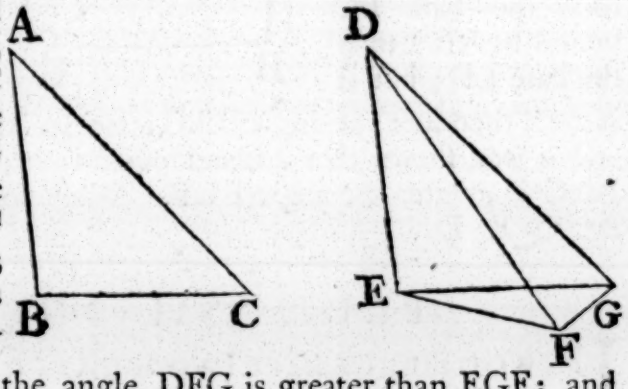
See N. **IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the angle contained by the two ſides of one of them greater than the angle contained by the two ſides equal to them, of the other; the baſe of that which has the greater angle ſhall be greater than the baſe of the other.**

Let ABC, DEF be two triangles which have the two ſides AB,

AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the baſe BC is alſo greater than the baſe EF. Book I.

Of the two ſides DE, DF, let DE be the ſide which is not greater than the other, and at the point D in the ſtraight line DE make [a] the angle EDG equal to the angle BAC; and make DG equal [b] to AC or DF, and join EG, GF. a. 23. 1. b. 3. 1.

Becauſe AB is equal to DE, and AC to DG, the two ſides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the baſe BC is equal [c] to the baſe EG; and becauſe DG is equal to DF, the angle DFG is equal [d] to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF; and becauſe the angle EFG of the triangle EFG is greater than the angle EGF, and that the greater [e] ſide is oppoſite to the greater angle; the ſide EG is therefore greater than the ſide EF; but EG is equal to BC; and therefore alſo BC is greater than EF. Therefore if two triangles, &c. Q. E. D.



c. 4. 1.

d. 5. 1.

e. 19. 1.

PROP. XXV. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the baſe of the one greater than the baſe of the other; the angle alſo contained by the ſides of that which has the greater baſe, ſhall be greater than the angle contained by the ſides equal to them, of the other.

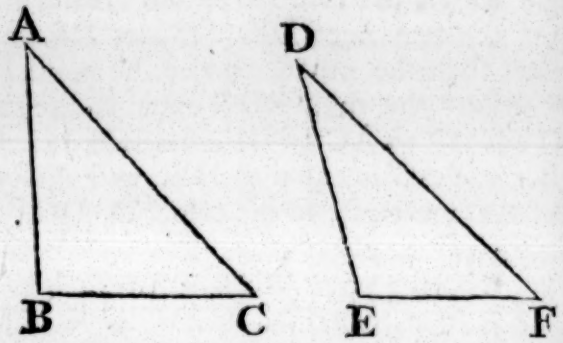
Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the baſe CB greater than the baſe EF; the angle BAC is likewiſe greater than the angle EDF.

Book I.

For, if it be not greater, it muſt either be equal to it, or leſs; but the angle BAC is not equal to the angle EDE, because then the baſe BC would be equal [a] to EF; but it is not; therefore the angle BAC is not equal to the angle EDF; neither is it leſs; becauſe then the baſe BC would be leſs [b] than the baſe EF; but it is not; therefore the angle BAC is not leſs than the angle EDF; and it was ſhewn that it is not equal to it; therefore the angle BAC is greater the angle EDF. Wherefore, if two triangles, &c. Q. E. D.

a. 4. 1.

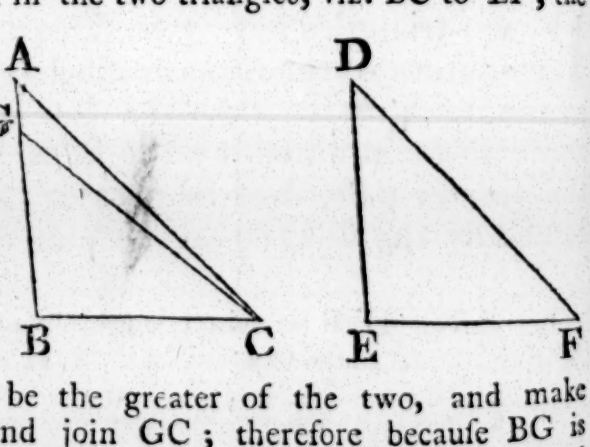
b 24. 1.



PROP. XXVI. THEOR.

IF two triangles have two angles of one equal to two angles of the other, each to each; and one ſide equal to one ſide, viz. either the ſides adjacent to the equal angles, or the ſides oppoſite to equal angles in each; then ſhall the other ſides be equal, each to each, and alſo the third angle of the one to the third angle of the other.

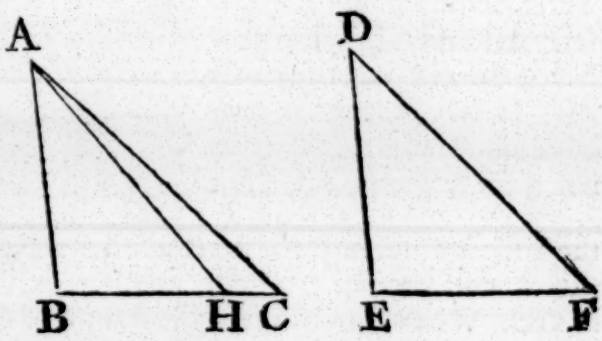
Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; alſo one ſide equal to one ſide; and firſt, let thoſe ſides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF; the other ſides ſhall be equal, each to each, viz. AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.



For, if AB be not equal to DE, one of them muſt be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore becauſe BG is equal

equal to DE, and BC to EF, the two ſides GB, BC, are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the baſe GC is equal [a] to the baſe DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal ſides are oppoſite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypotheſis, equal to the angle BCA; wherefore alſo the angle BCG is equal to the angle BCA, the leſs to the greater, which is impoſſible; therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; the baſe therefore AC is equal [a] to the baſe DF, and the third angle BAC to the third angle EDF.

a. 4. 1.



Next, let the ſides which are oppoſite to equal angles in each triangle be equal to one another, viz. AB to DE; likewiſe in this caſe, the other ſides ſhall be equal, AC to DF, and BC to EF; and alſo the third angle BAC to the third EDF.

For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH; and becauſe BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the baſe AH is equal to the baſe DF, and the triangle ABH to the triangle DEF, and the other angles ſhall be equal, each to each, to which the equal ſides are oppoſite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore alſo the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and oppoſite angle BCA; which is impoſſible [b]; wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the baſe AC is equal to the baſe DF, and the third angle BAC to the third angle EDF. Therefore if two triangles, &c. Q. E. D.

b. 16. 1.

PROP.

Book I.

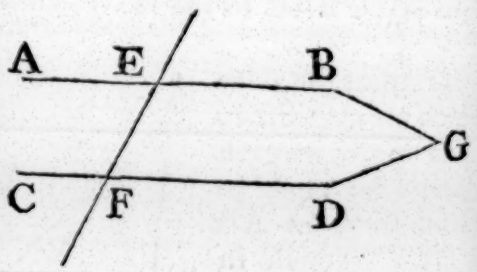
PROP. XXVII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the alternate angles equal to one another, theſe two ſtraight lines ſhall be parallel.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

For, if it be not parallel, AB and CD being produced ſhall meet either towards BD or towards AC; let them be produced and meet towards BD in the point G; therefore GEF is a tri-

a. 16. 1. angle, and its exterior angle AEF is greater [a] than the interior and oppoſite angle EFG; but it is alſo equal to it, which is impoſſible; therefore AB and CD being produced do not meet towards BD. In like manner it may be demonſtrated that they do not meet towards A,C; but



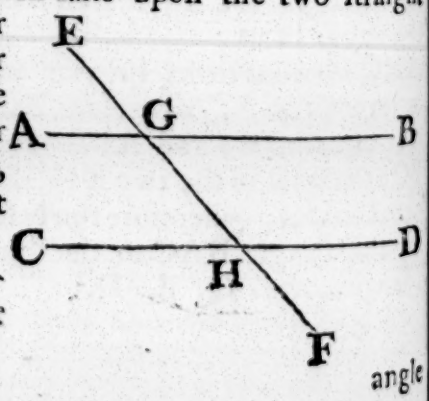
thoſe ſtraight lines which meet neither way, though produced ever

b. 35. Def. ſo far, are parallel [b] to one another. AB therefore is parallel to CD. Wherefore if a ſtraight line, &c. Q. E. D.

PROP. XXVIII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the exterior angle equal to the interior and oppoſite upon the ſame ſide of the line; or makes the interior angles upon the ſame ſide together equal to two right angles; the two ſtraight lines ſhall be parallel to one another.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the exterior angle EGB equal to the interior and oppoſite angle GHD upon the ſame ſide; or make the interior angles on the ſame ſide BGH, GHD together equal to two right angles; AB is parallel to CD.



Becauſe the angle EGB is equal to the angle GHD, and the

angle

gle EGB equal [a] to the angle AGH, the angle AGH is equal the angle GHD; and they are the alternate angles; therefore B is parallel [b] to CD; again, becauſe the angles BGH, GHD e equal [c] to two right angles, and that AGH, BGH are alſo ual [d] to two right angles; the angles AGH, BGH are equal the angles BGH, GHD: Take away the common angle GH, therefore the remaining angle AGH is equal to the reaining angle GHD; and they are alternate angles; therefore AB is parallel to CD. Wherefore if a ſtraight line, &c. E. D.

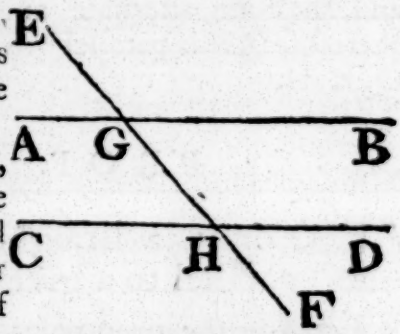
Book I.
a. 15. 1.
b. 27. 1.
c. By Hyp.
d. 13. 1.

PROP. XXIX. THEOR.

IF a ſtraight line falls upon two parallel ſtraight lines, it makes the alternate angles equal to one other; and the exterior angle equal to the interior d oppoſite upon the ſame ſide; and likewiſe the o interior angles upon the ſame ſide together e- al to two right angles.

See the notes on this propoſition.

Let the ſtraight line EF fall upon the parallel ſtraight lines B, CD; the alternate angles AGH, GHD are equal to one other; and the exterior angle EGB is equal to the interior d oppoſite, upon the ſame ſide HD; and the two interior angles GH, GHD upon the ſame ſide are ether equal to two right angles.



For, if AGH be not equal to GHD, e of them muſt be greater than the er; let AGH be the greater; and cauſe the angle AGH is greater n the angle GHD, add to each of m the angle BGH; therefore the angles AGH, BGH are ater than the angles BGH, GHD; but the angles AGH, GH are equal [a] to two right angles; therefore the angles GH, GHD are leſs than two right angles; but thoſe ſtraight s which, with another ſtraight line falling upon them, make interior angles on the ſame ſide leſs than two right angles, meet † together if continually produced; therefore the ight lines AB, CD, if produced far enough, ſhall meet; but y never meet, ſince they are parallel by the hypotheſis; refore the angle AGH is not unequal to the angle GHD, t is, it is equal to it; but the angle AGH is equal [b] to the le EGB; therefore likewiſe EGB is equal to GHD; add to each

a. 13. 1.
† 12. ax. See the notes on this propoſition.
b. 15. 1.

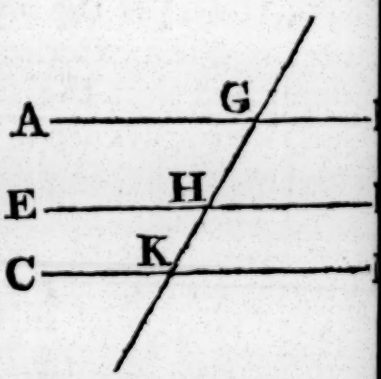
Book I. each of these the angle BGH; therefore the angles EGB, BG
are equal to the angles BGH, GHD; but EGB, BGH are e
c. 13. 1. qual [c] to two right angles; therefore also BGH, GHD are e
qual to two right angles. Wherefore if a straight line, &
Q. E. D.

PROP. XXX. THEOR.

STRAIGHT lines which are parallel to the sam straight line, are parallel to one another.

Let AB, CD be each of them parallel to EF; AB is also p rallel to CD.

Let the straight line GHK cut AB, EF, CD; and becau GHK cuts the parallel straight lines AB, EF, the angle AGH
a. 29. 1. is equal [a] to the angle GHF. Again, because the straight line GK cuts the parallel straight lines EF, CD, the angle GHF is equal [a] to the angle GKD; and it was shewn that the angle AGK is equal to the angle GHF; therefore also AGK is equal to GKD; and they are alternate angles;

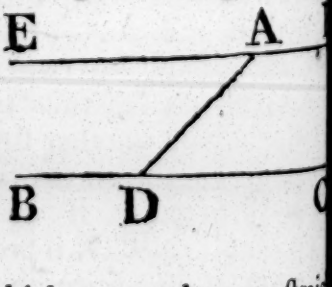


b. 27. 1. therefore AB is parallel [b] to CD. Wherefore straight lines, & Q. E D.

PROP. XXXI. PROB.

TO draw a straight line through a given poi parallel to a given straight line.

Let A be the given point, and BC the given straight line; is required to draw a straight line through the point A, parallel to the straight line BC.



In BC take any point D, and join AD; and at the point A in the
a. 23. 1. straight line AD make [a] the angle DAE equal to the angle ADC; and produce the straight line EA to F.

Because the straight line AD, which meets the two strai lines BC, EF, makes the alternate angles EAD, ADC equal one another, EF is parallel [b] to BC. Therefore the straight li
b. 27. 1. EA

AF is drawn through the given point A parallel to the given straight line BC. Which was to be done. Book. I.

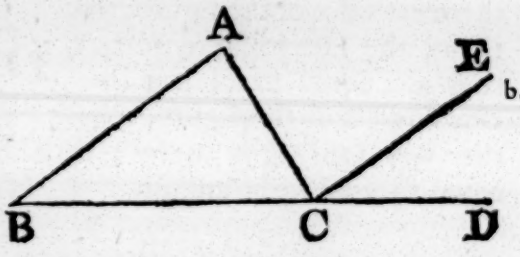
PROP. XXXII. THEOR.

IF a ſide of any triangle be produced, the exterior angle is equal to the two interior and oppoſite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its ſides BC be produced to D; the exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC; and the three interior angles of the triangle, viz. ABC, BCA, CAB are together equal to two right angles.

Through the point C draw CE parallel [a] to the ſtraight line AB; and becauſe AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal [b]. Again, becauſe AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and oppoſite angle ABC; but the angle ACE was ſhewn to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC; to theſe equals add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal [c] to two right angles; therefore alſo the angles CBA, BAC, ACB are equal to two right angles. Wherefore if a ſide of a triangle, &c. Q. E. D.

a. 31. 1.

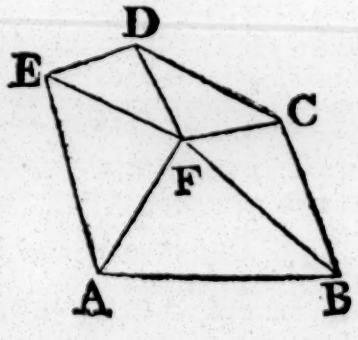


b. 29. 1.

c. 13. 1.

COR. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has ſides, by drawing ſtraight lines from a point F within the figure



to

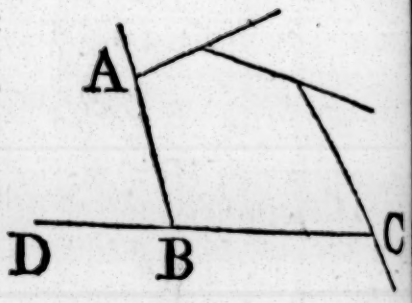
Book I. to each of its angles. And, by the preceeding Propoſition, the angles of theſe triangles are equal to twice as many right angles as there are triangles, that is, as there are ſides of the figure; and the ſame angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is [a], together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

a. 2. Cor. 15. 1.

Cor. 2. All the exterior angles of any rectilineal figure, are together equal to four right angles.

Becauſe every interior angle ABC with its adjacent exterior ABD is equal [b] to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are ſides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.

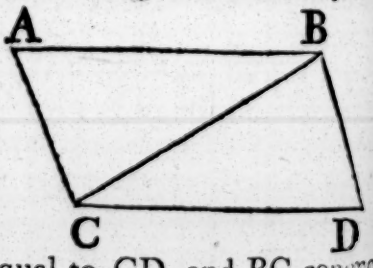
b. 13. 1.



PROP. XXXIII. THEOR.

THE ſtraight lines which join the extremities of two equal and parallel ſtraight lines, towards the ſame parts, are alſo themſelves equal and parallel.

Let AB, CD be equal and parallel ſtraight lines, and joined towards the ſame parts by the ſtraight lines AC, BD; AC, BD are alſo equal and parallel.



Join BC; and becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal [a]; and becauſe AB is equal to CD, and BC common to the two triangles ABC, DCB, the two ſides AB, BC are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; therefore the baſe AC is equal [b] to the baſe BD, and the triangle ABC to the triangle BCD, and the other angles to

a. 29 1.

b. 4. 1.

the other angles [b], each to each, to which the equal ſides are oppoſite; therefore the angle ACB is equal to the angle CBD; and becauſe the ſtraight line BC meets the two ſtraight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel [c] to BD; and it was ſhown to be equal to it. Therefore ſtraight lines, &c. Q. E. D.

Book I.
b. 4. 1.
c. 27. 1.

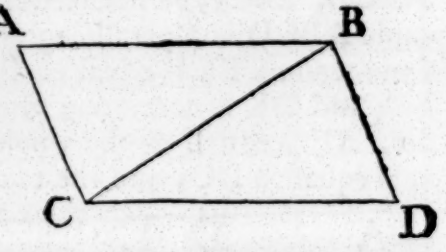
PROP. XXXIV. THEOR.

THE oppoſite ſides and angles of parallelograms are equal to one another, and the diameter biſects them, that is, divides them in two equal parts.

N. B. *A paralellogram is a four ſided figure, of which the oppoſite ſides are parallel; and the diameter is the ſtraight line joining two of its oppoſite angles.*

Let ABCD be a parallelogram, of which BC is a diameter; the oppoſite ſides and angles of the figure are equal to one another; and the diameter BC biſects it.

Becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal [a] to one another; and becauſe AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD, are equal [a] to one another; wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD the other, each to each, and one ſide BC common to the two triangles, which is adjacent to their equal angles; therefore their other ſides ſhall be equal, each to each, and the third angle of the one to the third angle of the other [b], viz. the ſide AB to the ſide CD, and AC to BD, and the angle BAC equal to the angle BDC: And becauſe the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB; the whole angle ABD is equal to the whole angle ACD: And the angle BAC has been ſhown to be equal to the angle BDC; therefore the oppoſite ſides and angles of parallelograms are equal to one another; alſo, their diameter biſects them; for AB being equal to CD, and BC common, the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC



a. 29. 1.
b. 26. 1.

is

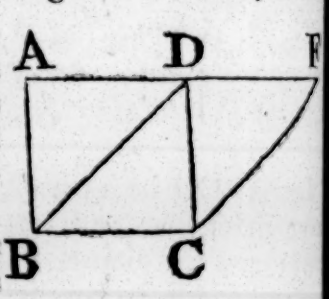
Book I. is equal to the angle BCD; therefore the triangle ABC is e
c. 4. 1. qual [c] to the triangle BCD, and the diameter BC divides th
parallelogram ACDB in two equal parts. Q. E. D.

PROP. XXXV. THEOR.

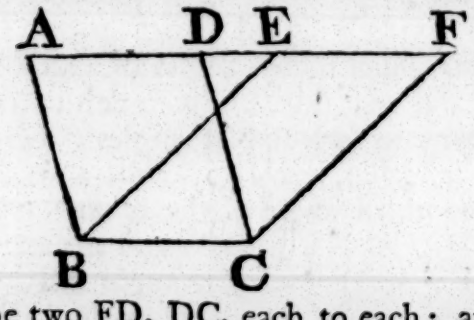
See N. PARALLELOGRAMS upon the ſame baſe and betwee
-the ſame parallels, are equal to one another.

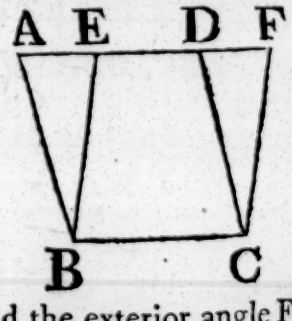
See the 2d and 3d figures. Let the parallelograms ABCD, EBCF, be upon the ſam
baſe BC, and between the ſame parallels AF, BC; the parallelo
gram ABCD ſhall be equal to the parallelogram EBCF.

If the ſides AD, DF of the parallelograms ABCD, DBC
oppoſite to the baſe BC be terminated in the ſame point D; it is plain that each of the parallelograms is
a 34. 1. double [a] of the triangle BDC; and they are therefore equal to one another.



But if the ſides AD, EF, oppoſite to the baſe BC of the parallelograms ABCD, EBCF, be not terminated in the ſame point; then be
cauſe ABCD is a parallelogram, AD, is equal [a] to BC; for th
b. 1. Ax. ſame reaſon, EF is equal to BC; wherefore AD is equal [b] t
EF; and DE is common; therefore the whole, or the remain
c. 2. or 3. Ax. der, AE is equal [c] to the whole, or the remainder DF; AB al
ſo is equal to DC; and the two EA, AB are therefore equal t





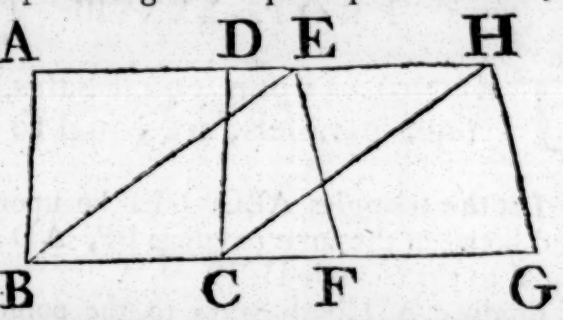
the two FD, DC, each to each; and the exterior angle FDC i
d. 29. 1. equal [d] to the interior EAB; therefore the baſe EB is equa
e. 4. 1. to the baſe FC, and the triangle EAB equal [e] to the triangl
FDC; take the triangle FDC from the trapezium ABCF, an
from the ſame trapezium take the triangle EAB; the remain
f. 3. Ax. ders therefore are equal [f], that is, the parallelogram ABCD i
equal to the parallelogram EBCF. Therefore parallelograms up
on the ſame baſe, &c. Q. E. D.

PROP.

PROP. XXXVI. THEOR.

PARALLELORGAMS upon equal baſes and between the ſame parallels, are equal to one another.

Let ABCD, EFGH be parallelograms upon equal baſes BC, FG, and between the ſame parallels AH, BG; the parallelogram ABCD is equal to EFGH.



Join BE, CH; and becauſe BC is equal to FG, and FG to[a] EH, BC is is equal to EH; and they are parallels, and joined towards the ſame parts by the ſtraight lines BE, CH: But ſtraight lines which join equal and parallel ſtraight lines towards the ſame parts, are themſelves equal and parallel[b]; therefore EB, CH are both equal and parallel, and EBCH is a parallelogram; and it is equal[c] to ABCD, becauſe it is upon the ſame baſe BC, and between the ſame parallels BC, AD: For the like reaſon the parallelogram EFGH is equal to the ſame EBCH: Therefore alſo the parallelogram ABCD is equal to EFGH. Wherefore parallelograms, &c. Q. E. D.

a. 34. 1.

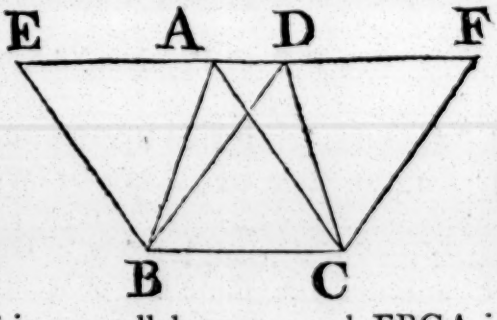
b. 33. 1.

c. 35. 1.

PROP. XXXVII. THEOR.

TRIANGLES upon the ſame baſe, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DBC be upon the ſame baſe BC and between the ſame parallels AD, BC: The triangle ABC is equal to the triangle DBC.



Produce AD both ways to the points E, F, and thro' B draw[a] BE parallel to CA; and thro' C draw CF parallel to BD: Therefore each of the figures EBCA, DBCF is a parallelogram; and EBCA is equal[b] to DBCF, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, EF; and the triangle ABC is

a. 31. 1.

b. 35. 1.

Book II. the half of the parallelogram EBCA, becauſe the diameter AB biſects it; and the triangle DBC is the half of the parallelogram DBCF, becauſe the diameter DC biſects it: But the halves of equal things are equal; therefore the triangle ABC is equal to the triangle DBC. Wherefore triangles, &c. Q. E. D.

c. 34. 1.

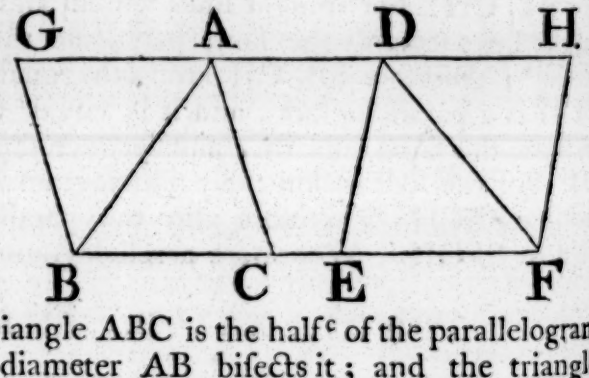
d. 7. Ax.

PROP. XXXVIII. THEOR.

TRIANGLES upon equal baſes, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal baſes BC, EF, and between the ſame parallels BF, AD: The triangle ABC is equal to the triangle DEF.

Produce AD both ways to the points G, H, and thro' B draw BG parrallel to CA, and thro' F draw FH parallel to ED: Then each of the figures GBCA, DEFH is a parallelogram; and they are equal to one another, becauſe they are upon equal baſes BC, EF and between the ſame parallels BF, GH; and the triangle ABC is the half of the parallelogram GBCA, becauſe the diameter AB biſects it; and the triangle DEF is the half of the parallelogram DEFH, becauſe the diameter DF biſects it: But the halves of equal things are equal; therefore the triangle ABC is equal to the triangle DEF. Wherefore triangles, &c. Q. E. D.



a. 31. 1.

b. 36. 1.

c. 34. 1.

d. 7. Ax.

PROP. XXXIX. THEOR.

EQUAL triangles upon the ſame baſe, and upon the ſame ſide of it, are between the ſame parallels.

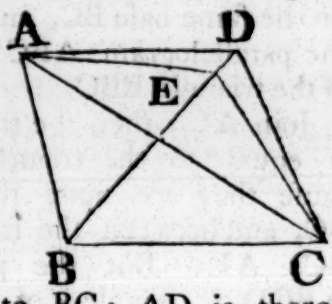
Let the equal triangles ABC, DBC be upon the ſame baſe BC, and upon the ſame ſide of it; they are between the ſame parallels.

Join AD; AD is parallel to BC; for, if it is not, thro' the point A draw AE parallel to BC, and join EC: The triangle

a. 31. 1.

Book I.

ngle ABC is equal [b] to the triangle EBC, becauſe it is upon he ſame baſe BC, and between the ſame arallels BC, AE: But the triangle ABC is equal to the triangle BDC; herefore alſo the triangle BDC is e- ual to the triangle EBC, the greater o the leſs, which is impoſſible: There- ore AE is not parallel to BC. In the me manner it can be demonſtrated hat no other line but AD is parallel to BC; AD is there- ore parallel to it. Wherefore equal triangles upon, &c. Q. E. D.



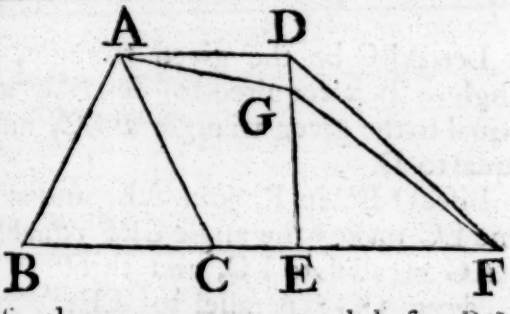
b. 37. 1.

PROP. XL. THEOR.

EQUAL triangles upon equal baſes, in the ſame ſtraight line, and towards the ſame parts, are etween the ſame parallels.

Let the equal triangles ABC, DEF be upon equal baſes BC, F, in the ſame ſtraight ne BF, and towards the me parts; they are between e ſame parallels.

Join AD; AD is paral- l to BC: For, if it is not, hro' A draw [a] AG paral- l to BF, and join GF: he triangle ABC is equal [b] o the triangle GEF, becauſe they are upon equal baſes BC, F, and between the ſame parallels BF, AG: But the triangle BC is equal to the triangle DEF; therefore alſo the triangle EF is equal to the triangle GEF, the greater to the leſs, which impoſſible: Therefore AG is not parallel to BF: And in the me manner it can be demonſtrated that there is no other paral- l to it but AD; AD is therefore parallel to BF. Wherefore qual triangles, &c. Q. E. D.



a. 31. 1.

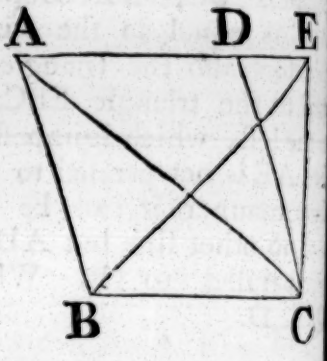
b. 38. 1.

PROP. XLI. THEOR.

IF a parallelogram and triangle be upon the ſame baſe, and between the ſame parallels; the paral- elogram ſhall be double of the triangle.

Book I.

Let the parallelogram ABCD and the triangle EBC be upon the ſame baſe BC, and between the ſame parallels BC, AE; the parallelogram ABCD is double of the triangle EBC.



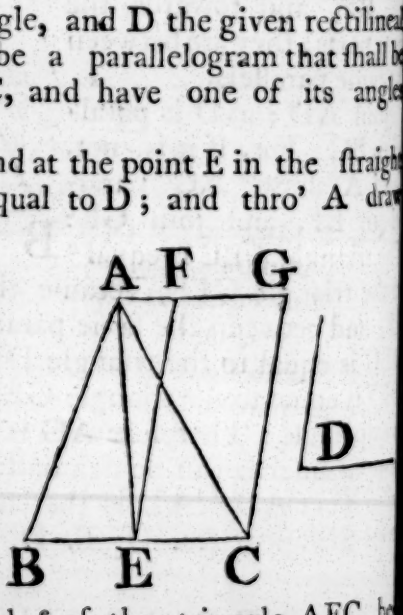
Join AC; then the triangle ABC
a. 37. 1. is equal [a] to the triangle EBC, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, AE: But the parallelogram
b. 34. 1. ABCD is double [b] of the triangle ABC, becauſe the diameter AC divides it into two equal parts; wherefore ABCD is alſo double of the triangle EBC. Therefore if a parallelogram, &c. Q. E. D.

PROP. XLII. PROB.

TO deſcribe a parallelogram that ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to deſcribe a parallelogram that ſhall be equal to the given triangle ABC, and have one of its angles equal to D.

a. 10. 1. Biſect [a] BC in E, join AE, and at the point E in the ſtraight
b. 23. 1. line EC make [b] the angle CEF equal to D; and thro' A draw
c. 31. 1. [c] AG parallel to EC, and thro' C draw CG [c] parallel to EF: Therefore FECG is a parallelogram: And becauſe BE is equal to EC, the triangle ABE is like-
d. 38. 1. wiſe equal [d] to the triangle AEC, ſince they are upon equal baſes BE, EC, and between the ſame parallels BC, AG; therefore the triangle ABC is double of the triangle AEC: And the paral-



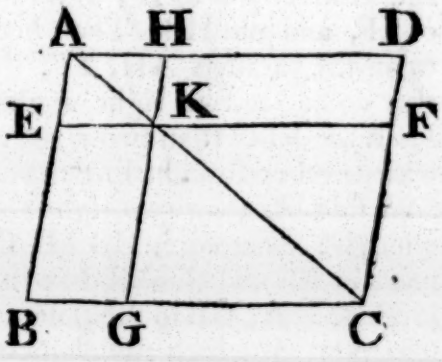
c. 41. 1. lelogram FECG is likewiſe double [c] of the triangle AEC, becauſe it is upon the ſame baſe, and between the ſame parallels: Therefore the parallelogram FECG is equal to the triangle ABC, and it has one of its angles CEF equal to the given angle D: Wherefore there has been deſcribed a parallelogram

logram FECG equal to a given triangle ABC, having one of its angles CEF equal to the given angle D. Which was to be done.

PROP. XLIII. THEOR.

THE complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diameter is AC, and EH, FG the parallelograms about AC, *that is, thro which* AC *paſſes*, and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefore called the complements: The complement BK is equal to the complement KD.



Becauſe ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal[a] to the triangle ADC: And becauſe EKHA is a parallelogram, the diameter of which is AK, the triangle AEK is equal to the triangle AHK: By the ſame reaſon, the triangle KGC is equal to the triangle KFC: Then, becauſe the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK together with the triangle KGC is equal to the triangle AHK together with the triangle KFC: But the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD. Wherefore the complements, &c. Q. E. D. [a] 34. 1.

PROP. XLIV. PROB.

TO a given ſtraight line to apply a parallelogram, which ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given ſtraight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the ſtraight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

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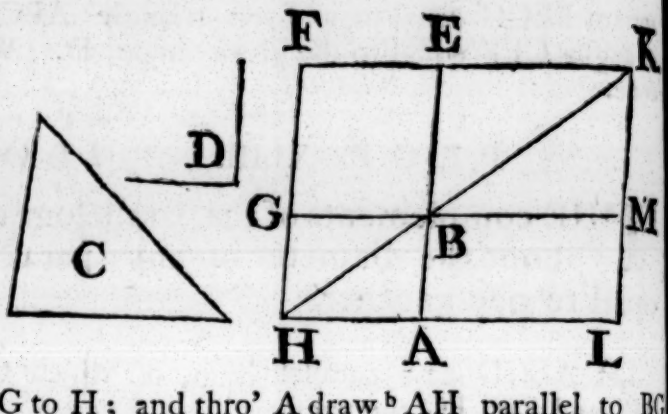
Make [a] the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, ſo that BE be in the ſame ſtraight line with AB,



a. 42. 1.

and produce FG to H; and thro' A draw [b] AH parallel to BG or EF, and join HB. Then becauſe the ſtraight line HF falls upon the parallels AH, EF, the angles AHF, HFE are toge-

b. 31. 1.

ther equal [c] to two right angles; wherefore the angles BHF, HFE are leſſer than two right angles: But ſtraight lines which with another ſtraight line make the interior angles upon the

c. 29. 1.

ſame ſide leſs than two right angles, do meet [d] if produced far enough: Therefore, HB FE ſhall meet, if produced; let them meet in K, and thro' K draw KL parallel to EA or FH, and produce HA, GB to the points L, M: Then HLKF is a parallelogram, of which the diameter is HK, and AG, ME are the

d. 12. Ax.

parallelograms about HK; and LB, BF are the complements; therefore LB is equal [e] to BF: But BF is equal to the triangle C; wherefore LB is equal to the triangle C: And becauſe the angle

e. 43. 1.

GBE is equal [f] to the angle ABM, and likewiſe to the angle D; the angle ABM is equal to the angle D: Therefore the parallelogram LB is applied to the ſtraight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D: Which was to be done.

f. 15. 1.

PROP. XLV. PROB.

TO deſcribe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to deſcribe a parallelogram equal to ABCD, and having an angle equal to E.

Join DB, and deſcribe [a] the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E;

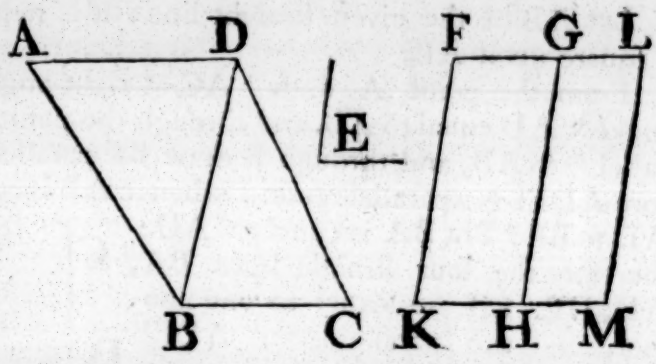
a. 42. 1.

and to the ſtraight line GH apply [b] the parallelogram GM equal

b. 44. 1.

to

to the triangle DBC, having the angle GHM equal to the angle E; and becauſe the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add to each of theſe the angle KHG; therefore the angles FKH, KHG are equal to the angles KHG, GHM; but FKH, KHG are equal [c] to two right angles; therefore alſo KHG, GHM are equal to two right angles; and becauſe at the point H in the ſtraight line GH, the two ſtraight lines KH, HM upon the oppoſite ſides of it make the adjacent angles equal to two right angles, KH is in the ſame ſtraight line [d] with HM; and becauſe the ſtraight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal [c]: Add to each of theſe, the angle HGL: Therefore the angles MHG, HGL are equal to the angles HGF, HGL: But the angles MHG, HGL are equal [c] to two right angles; wherefore alſo the angles HGF, HGL are equal to two right angles, and FG is therefore in the ſame ſtraight line with GL: And becauſe KF is parallel to HG, and HG to ML; KF is parallel [e] to ML: and KM, FL are parallels; wherefore KFLM is a parallelogram; and becauſe the triangle ABD is equal to the parallelogram HF, and the the triangle DBC to the parallelogram GM; the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM; therefore the parallelogram KFLM has been deſcribed equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E: Which was to be done.



c. 29. 1.

d. 14. 1.

c. 30. 1.

Cor. From this it is manifeſt how to a given ſtraight line to apply a parallelogram, which ſhall have an angle equal to a given rectilineal angle, and ſhall be equal to a given rectilineal figure, viz. by applying [b] to the given ſtraight line, a parallelogram equal to the firſt triangle ABD, and having an angle equal to the given angle.

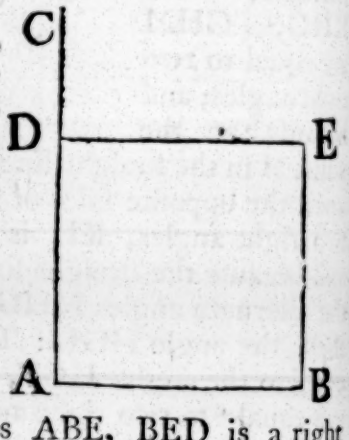
b. 44. 1.

PROP. XLVI. PROB.

TO deſcribe a ſquare upon a given ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe a ſquare upon AB.

a. 11. 1. From the point A draw [a] AC at right angles to AB; and
b. 3. 1. make [b] AD equal to AB, and through the point D draw DE pa-
c. 31. 1. rallel [c] to AB, and through B draw BE parallel to AD; there-
d. 34. 1. fore ADEB is a parallelogram; whence AB is equal [d] to DE, and AD to BE: But BA is equal to AD; therefore the four ſtraight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral, likewiſe all its angles are right angles; becauſe the ſtraight line AD meeting the parallels AB, DE, the angles BAD, ADE, are e-
c. 29. 1. qual [c] to two right angles; but BAD is a right angle, therefore alſo ADE is a right angle; but the oppoſite angles of parallelograms are equal [d]; therefore each of the oppoſite angles ABE, BED is a right angle; wherefore the figure ADEB is rectangular, and it has been demonſtrated that it is equilateral; it is therefore a ſquare, and it is deſcribed upon the given ſtraight line AB: Which was to be done



COR. Hence every parallelogram that has one right angle has all its angles right angles.

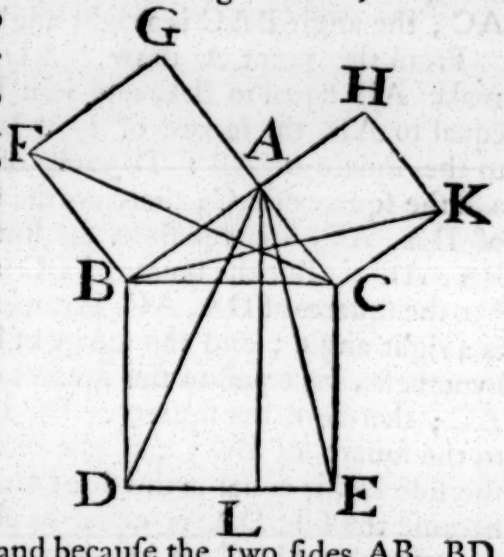
PROP. XLVII. THEOR.

IN any right angled triangle, the ſquare which is deſcribed upon the ſide ſubtending the right angle, is equal to the ſquares deſcribed upon the ſides which contain the right angle.

Let ABC be a right angled triangle having the right angle BAC; the ſquare deſcribed upon the ſide BC is equal to the ſquares deſcribed upon BA, AC.

a. 46. 1. On BC deſcribe [a] the ſquare BDEC, and on BA, AC the ſquares GB,

B, HC; and through A draw [b] AL parallel to BD or CE, and in AD, FC; then, because each of the angles BAC, BAG is a right angle [c], the two straight lines AC, AG upon the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the same straight line [d] with AG; for the same reason, AB and AH are in the same straight line; and because the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC, and the whole angle DBA is equal [e] to the whole FBC; and because the two sides AB, BD are equal to the two FB, BC, each to each, and the angle DBA equal to the angle FBC; therefore the base AD is equal [f] to the base FC, and the triangle ABD to the triangle FBC: Now the parallelogram BL is double [g] of the triangle ABD, because they are upon the same base BD, and between the same parallels BD, AL; and the square GB is double of the triangle FBC, because these also are upon the same base FB, and between the same parallels FB, GC: But the doubles of equals are equal [h] to one another: Therefore the parallelogram BL is equal to the square GB: And in the same manner, by joining AE, BK, it is demonstrated that the parallelogram CL is equal to the square HC: Therefore the whole square BDEC is equal to the two squares GB, HC; and the square BDEC is described upon the straight line BC, and the squares GB, HC upon BA, AC: Wherefore the square upon the side BC is equal to the squares upon the sides BA, AC. Therefore in any right angled triangle, &c. Q. E. D.



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b. 31. 1.

c. 30. def.

d. 14. 1.

e. 2. Ax.

f. 4. 1.

g. 41. 1.

h. 6. Ax.

PROP. XLVIII. THEOR.

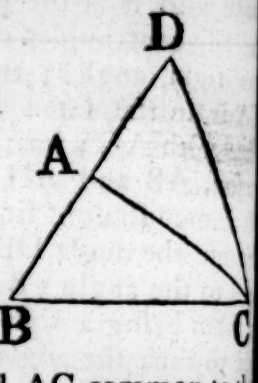
IF the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

If

Book I.

If the ſquare deſcribed upon BC, one of the ſides of the triangle ABC, be equal to the ſquares upon the other ſides BA, AC; the angle BAC is a right angle.

a. 11. 1. From the point A draw [a] AD at right angles to AC, and make AD equal to BA, and join DC: Then, becauſe DA is equal to AB, the ſquare of DA is equal to the ſquare of AB: To each of theſe add the ſquare of AC; therefore the ſquares of DA, AC, are equal to the ſquares of BA, AC: But the ſquare of DC is equal



b. 47. 1. [b] to the ſquares of DA, AC, becauſe DAC is a right angle; and the ſquare of BC, by hypotheſis, is equal to the ſquares of BA, AC; therefore the ſquare of DC is equal to the ſquare of BC; and therefore alſo the ſide DC is equal to the ſide BC. And becauſe the ſide DA is equal to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC; and the baſe DC is equal to the baſe BC; there-

c. 8. 1. fore the angle DAC is equal [c] to to the angle BAC: But DAC is a right angle, therefore alſo BAC is a right angle. Therefore if the ſquare, &c. Q. E. D.

THE ELEMENTS OF EUCLID.

BOOK II.

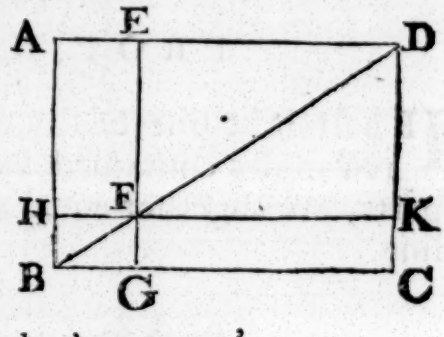
DEFINITIONS.

I.

EVERY right angled parallelogram is ſaid to be contained by any two of the ſtraight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. 'Thus the 'parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expreſſed by the letters 'AGK, or EHC which are 'at the oppoſite angles of 'the parallelograms which make the gnomon.'



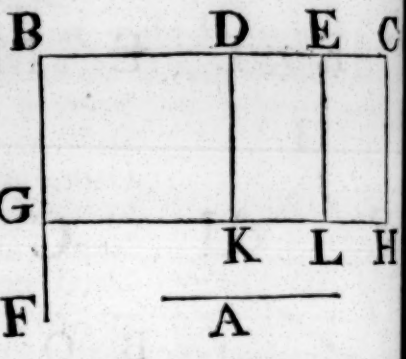
PROP. I. THEOR.

IF there be two ſtraight lines, one of which is divided into any number of parts; the rectangle contained by the two ſtraight lines, is equal to the rectangles contained by the undivided line, and the ſeveral parts of the divided line.

Let

Book II.

Let A and BC be two ſtraight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the ſtraight lines A, BC is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.



a. 11. 1. From the point B draw [a] BF at right angles to BC, and make BG equal [b] to A; and through G draw [c] GH parallel to BC; and through D, E, C draw [c] DK EL, CH parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, becauſe DK, that is, [d] BG, is equal to A; and in like manner the rectangle EH is contained by A, EC. Therefore the rectangle contained by A, BC is equal to the ſeveral rectangles contained by A, BD, and by A, DE; and by A, EC. Wherefore, if there be two ſtraight lines, &c. Q. E. D.

b. 3. 1.

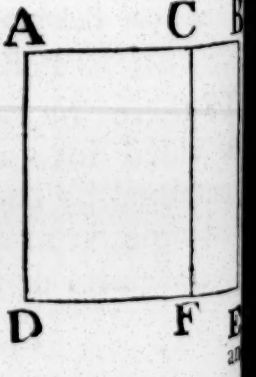
c. 31. 1.

d. 34. 1.

PROP. II. THEOR.

IF a ſtraight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the ſquare of the whole line.

Let the ſtraight line AB be divided into into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle † AB, AC, ſhall be equal to the ſquare of AB.



a. 46. 1. Upon AB deſcribe [a] the ſquare ADEB, and through C draw [b] CF, parallel to AD, or BE; then AE is equal to the rectangles AF, CE; and AE is the ſquare of AB;

b. 31. 1.

† N. B. To avoid repeating the word *contained* too frequently, the rectangle contained by two ſtraight lines AB, AC is ſometimes ſimply called the rectangle AB, AC.

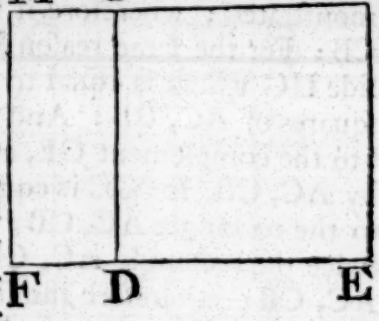
and AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB; therefore the rectangle contained by AB, AC, together with the rectangle AB, BC, is equal to the square of AB. If therefore a straight line, &c. Q. E. D. Book. II.

PROP. III. THEOR.

IF a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the foresaid part.

Let the straight line AB be divided into any two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB together with the square of BC.

Upon BC describe [a] the square CDEB, and produce ED to F, and through A draw [b] AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE; and AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and DB is the square of BC; therefore the rectangle AB, BC is equal to the rectangle AC, CB together with the square of BC. If therefore a straight line, &c. Q. E D.



a. 46. 1.

b. 31. 1.

PROP. IV. THEOR.

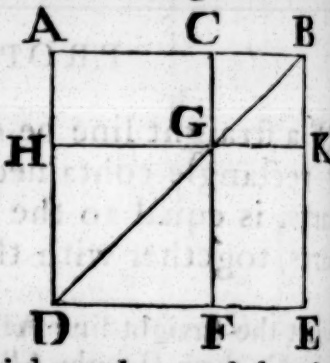
IF a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB and to twice the rectangle contained by AC, CB.

Upon

Book II.

Upon AB deſcribe [a] the ſquare ADEB, and join BD, and thro' C draw [b] CGF parallel to AD or BE, and thro' G draw HK parallel to AB or DE: And becauſe CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal [c] to the interior and oppoſite angle ADB; but ADB is equal [d] to the angle ABD, becauſe BA is equal to AD, being ſides of a ſquare; wherefore the angle CGB is equal to the angle GBC; and therefore [e] the ſide BC is equal to the ſide CG: But CB is equal [f] alſo to GK, and CG to BK; wherefore the figure CGKB is equilateral: It is likewiſe rectangular; for CG is parallel to BK, and CB meets them, the angles KBC, GCB are therefore equal to two right angles; and KBC is a right angle; wherefore GCB is a right angle; and therefore alſo the angles [f] CGK, GKB oppoſite to theſe are right angles, and CGKB is rectangular: But it is alſo equilateral, as was demonſtrated; wherefore it is a ſquare, and it is upon the ſide CB: For the ſame reaſon HF alſo is a ſquare, and it is upon the ſide HG which is equal to AC: Therefore HF, CK are the ſquares of AC, CB: And becauſe the complement AG is equal [g] to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; therefore GE is alſo equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB: And HF CK are the ſquares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the ſquares of AC, CB and to twice the rectangle AC, CB: But HF, CK, AG, GE make up the whole figure ADEB which is the ſquare of AB: Therefore the ſquare of AB is equal to the ſquares of AC, CB and twice the rectangle AC, CB. Wherefore if a ſtraight line, &c. Q. E. D.



a. 46. 1.
b. 31. 1.
c. 29. 1.
d. 5. 1.
e. 6. 1.
f. 34. 1.
g. 43. 1.

Cor. From the demonſtration it is manifeſt, that the parallelograms about the diameter of a ſquare are likewiſe ſquares.

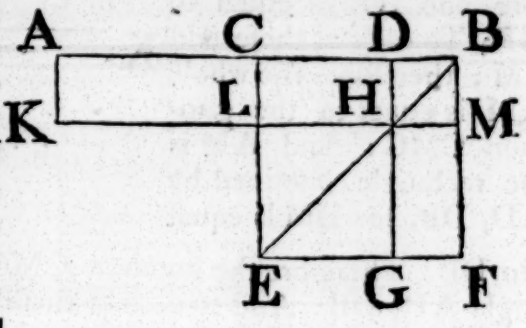
PROP.

PROP. V. THEOR.

a ſtraight line be divided into two equal parts, and alſo into two unequal parts; the rectangle ntained by the unequal parts, together with the are of the line between the points of ſection, is al to the ſquare of half the line.

et the ſtraight line AB be divided into two equal parts in point C, and into two unequal parts at the point D; the angle AD, DB together with the ſquare of CD, is equal e ſquare of CB.

Upon CB deſcribe [a] the ſquare CEFB, join BE, and thro' draw [b] DHG parallel to CE or BF; and thro' H draw M parallel to CB or EF; and alſo thro' A draw AK pal to CL or BM: And becauſe the complement CH is e- [c] to the complement HF, to each of theſe add DM; efore the whole CM qual to the whole DF; CM is equal [d] to AL, auſe AC is equal to ; therefore alſo AL is l to DF. To each of add CH, and the le AH is equal to and CH: But AH is rectangle contained by , DB, for DH is equal DB; and DF together with CH is the gnomon CMG; efore the gnomon CMG is equal to the rectangle AD, DB: ach of theſe add LG, which is equal [e] to the ſquare of CD, efore the gnomon CMG together with LG is equal to the angle AD, DB together with the ſquare of CD: But the mon CMG and LG make up the whole figure CEFB, which e ſquare of CB: Therefore the rectangle AD, DB together the ſquare of CD is equal to the ſquare of CB. Wherefore ſtraight line, &c. Q. E. D.



a. 46. 1.

b. 31. 1.

c. 43. 1.

d. 36. 1.

e. Cor. 4. 2.

PROP.

Book II.

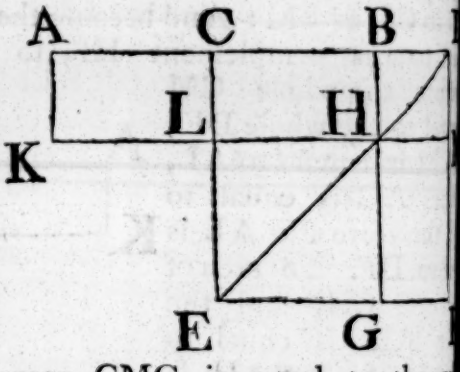
PROP. VI. THEOR.

IF a ſtraight line be biſected, and produced to a point; the rectangle contained by the whole li thus produced, and the part of it produced, togeth with the ſquare of half of the line biſected, is equ to the ſquare of the ſtraight line which is made of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced the point D; the rectangle AD, DB, together with the ſqua of CB, is equal to the ſquare of CD.

a. 46. 1. Upon CD deſcribe [a] the ſquare CEFD, join DE, and th

b. 31. 1. B draw [b] BHG parallel to CE or DF, and thro' H draw KL parallel to AD or EF, and alſo thro' A draw AK parallel CL or DM: And becauſe AC is equal to CB, the rectan



e. 36. 1. AL is equal [c] to CH; but

d. 43. 1. CH is equal [d] to HF; therefore alſo AL is equal to HF: To each of theſe add CM; therefore the whole AM is equal to the gnomon CMG: And AM is the rectangle contained by AD, DB, for DM is equal

c. Cor. 4. 2. [e] to DB: Therefore the gnomon CMG is equal to the re angle AD, DB: Add to each of theſe LG, which is equal the ſquare of CB; therefore the rectangle AD, DB, toge with the ſquare of CB, is equal to the gnomon CMG and figure LG: But the gnomon CMG and LG make up whole figure CEFD, which is the ſquare of CD; there the rectangle AD, DB, together with the ſquare of CB, is qual to the ſquare of CD. Wherefore if a ſtraight line, Q. E. D.

PROP. VII. THEOR.

IF a ſtraight line be divided into any two parts, t ſquares of the whole line, and of one of the parts, a equal to twice the rectangle contained by the wh and that part, together with the ſquare of the ot part.

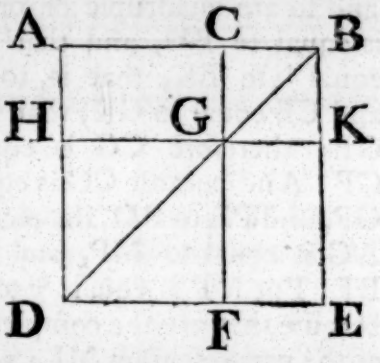
Let the ſtraight line AB be divided into any two parts

he point C; the ſquares of AB, BC are equal to twice the rectangle AB, BC together with the ſquare of AC.

Upon AB deſcribe [a] the ſquare ADEB, and conſtruct the figure as in the preceeding Propoſitions: And becauſe AG is equal [b] to GE, add to each of them CK, the whole AK is therefore equal to the whole CE; therefore AK, CE are double of AK: But AK, CE are the gnomon AKF together with the ſquare CK; therefore the gnomon AKF together with the ſquare CK is double of AK: But twice the rectangle AB, BC is double of AK, for BK is equal [c] to BC: Therefore the gnomon AKF, together with the ſquare CK, is equal to twice the rectangle AB, BC: To each of theſe equals add HF, which is equal to the ſquare of AC; therefore the gnomon AKF, together with the ſquares CK, HF, is equal to twice the rectangle AB, BC and the ſquare of AC: But the gnomon AKF, together with the ſquares CK, HF, make up the whole figure ADEB and CK, which are the ſquares of AB and BC: Therefore the ſquares of AB and BC are equal to twice the rectangle AB, BC together with the ſquare of AC. Wherefore, if a ſtraight line, &c. Q. E. D.

a. 46. 1.

b. 43. 1.



c. Cor. 4. 2.

PROP. VIII. THEOR.

IF a ſtraight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the ſquare of the other part, is equal to the ſquare of the ſtraight line which is made up of the whole and that part.

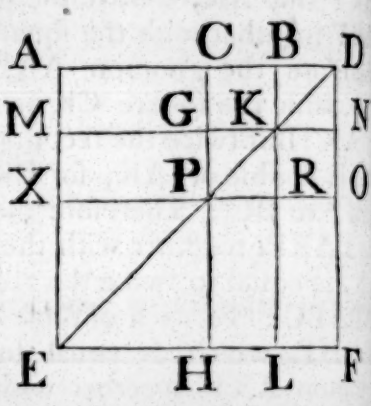
Let the ſtraight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of the ſtraight line made up of AB and BC together.

Produce AB to D, ſo that BD be equal to CB, and upon AD deſcribe the ſquare AEFD; and conſtruct two figures ſuch as in the preceeding. Becauſe CB is equal to BD, and that CB is equal [a] to GK, and BD to KN; therefore GK is

a. 34. 1.

Book II

equal to KN: For the ſame reaſon, PR is equal to RO; and becauſe CB is equal to BD, and GK to KN, the rectangle CK is equal [b] to BN, and GR to RN: But CK is equal [c] to RN, becauſe they are the complements of the parallelogram CO; therefore alſo BN is equal to GR: And the four rectangles BN, CK, GR, RN, are therefore equal to one another, and ſo are quadruple of one of them CK: Again, becauſe CB is equal to BD, and that BD is equal [d] to BK, that is, to CG; and CB equal to GK, that [d] is, to GP; therefore CG is equal to GP: And becauſe CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF: But MP is equal [e] to PL, becauſe they are the complements of the parallelogram ML; wherefore AG alſo is equal to RF: Therefore the four rectangles AG, MP, PL, RF are equal to one another, and ſo are quadruple of one of them AG. And it was demonſtrated, that the four CK, BN, GR, RN are quadruple of CK: Therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK: And becauſe AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC is quadruple of AK: But the gnomon AOH was demonſtrated to be quadruple of AK; therefore four times the rectangle AB, BC is equal to the gnomon AOH. To each of theſe add XH, which is equal [d] to the ſquare of AC; therefore four times the rectangle AB, BC, together with the ſquare of AC is equal to the gnomon AOH and the ſquare XH: But the gnomon AOH and XH make up the figure AEFD which is the ſquare of AD: Therefore four times the rectangle AB, BC together with the ſquare of AC is equal to the ſquare of AD, that is, of AB and BC added together in one ſtraight line. Wherefore, if a ſtraight line, &c. Q. E. D.



b. 36. 1.
c. 43. 1.
d. Cor. 4. 2.
e. 43. 1.
d. Cor. 4. 2.

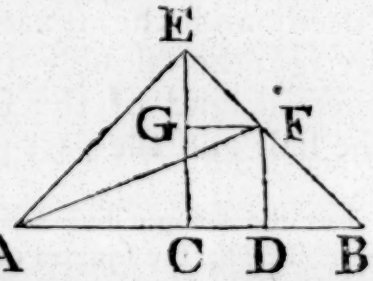
PROP.

PROP. IX. THEOR.

IF a ſtraight line be divided into two equal, and alſo into two unequal parts; the ſquares of the two unequal parts are together double of the ſquare of half the line, and of the ſquare of the line between the points of ſection.

Let the ſtraight line AB be divided at the point C into two equal, and at D into two unequal parts: The ſquares of AD, DB are together double of the ſquares of AC, CD.

From the point C draw [a] CE at right angles to AB, and make it equal to AC or CB, and join EA, EB; through D draw [b] DF parallel to CE, and through F draw FG parallel to AB; and join AF: Then, becauſe AC is equal to CE, the angle EAC is equal [c] to the angle AEC; and becauſe the angle ACE is a right angle, the two others AEC, EAC together make one right angle [d]; and they are equal to one another; each of them therefore is half of a right angle. For the ſame reaſon each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle: And becauſe the angle GEF is half a right angle, and EGF a right angle, for it is equal [e] to the interior and oppoſite angle ECB, the remaining angle EFG is half a right angle; therefore the angle GEF is equal to the angle EFG, and the ſide EG equal [f] to the ſide GF: Again, becauſe the angle at B is half a right angle, and FDB a right angle, for it is equal [e] to the interior and oppoſite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the ſide DF to [f] the ſide DB: And becauſe AC is equal to CE, the ſquare of AC is equal to the ſquare of CE; therefore the ſquares of AC, CE are double of the ſquare of AC: But the ſquare of EA is equal [g] to the ſquares of AC, CE, becauſe ACE is a right angle; therefore the ſquare of EA is double of the ſquare of AC: Again, becauſe EG is equal to GF, the ſquare of EG is equal to the ſquare of GF; therefore the ſquares of EG, GF are double of



a. 11. 1.
b. 31. 1.
c. 5. 1.
d. 32. 1.
e. 29. 1.
f. 6. 1.
g. 47. 1.

Book II. the ſquare of GF; but the ſquare of EF is equal to the ſquares of EG, GF; therefore the ſquare of EF is double of the ſquare
h. 34. 1. GF: And GF is equal [h] to CD; therefore the ſquare of EF is double of the ſquare of CD: But the ſquare of AE is likewiſe double of the ſquare of AC; therefore the ſquares of AE, EF are double of the ſquares of AC, CD: And the ſquare of AF is
i. 47. 1. equal [i] to the ſquares of AE, EF, becauſe AEF is a right angle; therefore the ſquare of AF is double of the ſquares of AC, CD: But the ſquares of AD, DF are equal [i] to the ſquare of AF, becauſe the angle ADF is a right angle; therefore the ſquares of AD, DF are double of the ſquares of AC, CD: And DF is equal to DB; therefore the ſquares of AD, DB, are double of the ſquares of AC, CD. If therefore a ſtraight line, &c. Q. E. D.

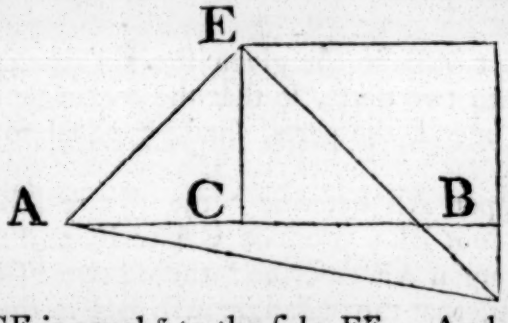
PROP. X. THEOR.

IF a ſtraight line be biſected, and produced to any point, the ſquare of the whole line thus produced, and the ſquare of the part of it produced, are together double of the ſquare of half the line biſected, and of the ſquare of the line made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the ſquares of AD, DB are double of the ſquares of AC, CD.

a. 11. 1. From the point C draw [a] CE at right angles to AB: And make it equal to AC or CB, and join AE, EB; through E draw
b. 31. 1. [b] EF parallel to AB, and through D draw DF parallel to CE: And becauſe the ſtraight line EF meets the parallels EC, FD, the
c. 29. 1. angles CEF, EFD are equal [c] to two right angles; and therefore the angles BEF, EFD are leſs than two right angles: But ſtraight lines which with another ſtraight line make the interior angles
d. 12. Ax. upon the ſame ſide leſs than two right angles, do meet [d] if produced far enough: Therefore EB, FD ſhall meet, if produced, toward BD: Let them meet in G, and join AG: Then, becauſe
e. 5. 1. AC is equal to CE, the angle CEA is equal [e] to the angle EAC; and the angle ACE is a right angle; therefore each of the
* 32. 1. angles CEA, EAC is half a right angle *: For the ſame reaſon, each

each of the angles CEB, EBC is half a right angle; therefore AEB is a right angle: And becauſe EBC is half a right angle, DBG is alſo [f] half a right angle, for they are vertically oppoſite; but BDG is a right angle, becauſe it is equal [e] to the alternate angle DCE; therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG; wherefore alſo the ſide BD is equal [g] to the ſide DG: Again, becauſe EGF is half a right angle, and that the angle at F is a right angle, becauſe it is equal [h] to the oppoſite angle ECD, the remaining angle FEG is half a right angle, and equal to the angle EGF; wherefore alſo the ſide GF is equal [g] to the ſide FE. And becauſe EC is equal to CA, the ſquare of EC is equal to the ſquare of CA; therefore the ſquares of EC, CA are double of the ſquare of CA: But the ſquare of EA is equal [i] to the ſquares of EC, CA; therefore the ſquare of EA is double of the ſquare of AC: Again, becauſe GF is equal to FE, the ſquare of GF is equal to the ſquare of FE; and therefore the ſquares of GF, FE are double of the ſquare of EF: But the ſquare of EG is equal [i] to the ſquares of GF, FE; therefore the ſquare of EG is double of the ſquare of EF: And EF is equal to CD; wherefore the ſquare of EG is double of the ſquare of CD: But it was demonſtrated, that the ſquare of EA is double of the ſquare of AC; therefore the ſquares of AE, EG are double of the ſquares of AC, CD: And the ſquare of AG is equal [i] to the ſquares of AE, EG; therefore the ſquare of AG is double of the ſquares of AC, CD: But the ſquares of AD, DG are equal [i] to the ſquare of AG; therefore the ſquares of AD, DG are double of the ſquares of AC, CD: But DG is equal to DB; therefore the ſquares of AD, DB are double of the ſquares of AC, CD: Wherefore, if a ſtraight line, &c. Q. E. D.



Book II.

[f] 15. 1.
[e] 29. 1.
[g] 6. 1.
[h] 34. 1.
[i] 47. 1.

PROP. XI. PROB.

TO divide a given ſtraight line into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

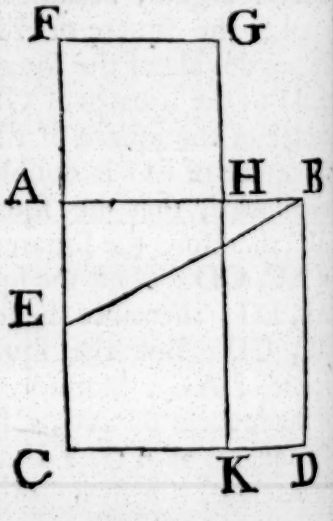
Let AB be the given ſtraight line; it is required to divide it into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

a. 46. 1. b. 10. 1. c. 3. 1. Upon AB deſcribe [a] the ſquare ABDC, biſect [b] AC in E, and join BE; produce CA to F, and make [c] EF equal to FB; and upon AF deſcribe [a] the ſquare FGHA; AB is divided in H ſo, that the rectangle AB, BH is equal to the ſquare of AH.

Produce GH to K: Becauſe the ſtraight line AC is biſected in E, and produced to the point F, the rectangle CF, FA, together with the ſquare of AE, is equal [d] to the ſquare of EF: But EF is equal to EB; therefore the rectangle CF, FA, together with the ſquare of AE, is equal to the ſquare of EB: And the ſquares of BA, AE are equal [e] to the ſquare of EB, becauſe the angle EAB is a right angle; therefore the rectangle CF, FA, together with the ſquare of AE, is equal to the ſquares of BA, AE: Take away the ſquare of AE, which is common to both, therefore the remaining rectangle CF, FA is equal to the ſquare of AB: And the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the ſquare of AB; therefore FK is equal to AD: Take away the common part AK, and the remainder FH is equal to the remainder HD: And HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the ſquare of AH: Therefore the rectangle AB, BH is equal to the ſquare of AH: Wherefore the ſtraight line AB is divided in H, ſo that the rectangle AB, BH is equal to the ſquare of AH. Which was to be done.

d. 6. 2.

e. 47. 1.



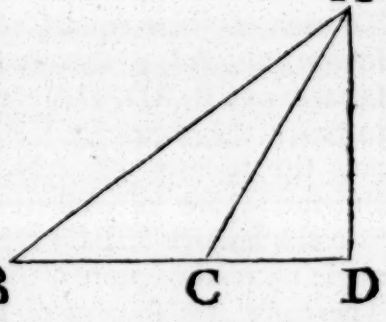
PROP.

PROP. XII. THEOR.

IN obtuſe angled triangles, if a perpendicular be drawn from any of the acute angles to the oppoſite ſide produced, the ſquare of the ſide ſubtending the obtuſe angle, is greater than the ſquares of the ſides containing the obtuſe angle, by twice the rectangle contained by the ſide upon which, when produced, the perpendicular falls, and the ſtraight line intercepted without the triangle between the perpendicular and the obtuſe angle.

Let ABC be an obtuſe angled triangle, having the obtuſe angle ACB, and from the point A let AD be drawn [a] perpendicular to BC produced: The ſquare of AB is greater than the ſquares of AC, CB by twice the rectangle BC, CD. a. 12. 1.

Becauſe the ſtraight line BD is divided into two parts in the point C, the ſquare of BD is equal [b] to the ſquares of BC, CD, and twice the rectangle BC, CD: To each of theſe equals add the ſquare of DA; and the ſquares of BD, DA are equal to the ſquares of BC, CD, DA, and twice the rectangle BC, CD: But the ſquare of BA is equal [c] to the ſquares of BD, DA, becauſe the angle at D is a right angle; and the ſquare of CA is equal [c] to the ſquares of CD, DA: Therefore the ſquare of BA is equal to the ſquares of BC, CA, and twice the rectangle BC, CD; that is, the ſquare of BA is greater than the ſquares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuſe angled triangles, &c. Q. E. D.



b. 4. 2.

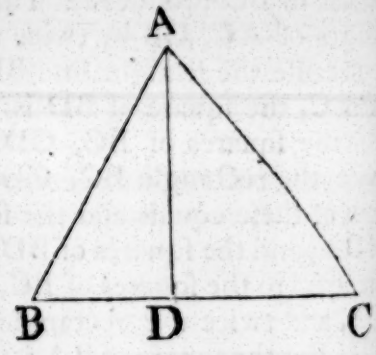
c. 47. 1.

PROP. XIII. THEOR.

See N. IN every triangle, the ſquare of the ſide ſubtending any of the acute angles, is leſs than the ſquares of the ſides containing that angle, by twice the rectangle contained by either of theſe ſides, and the ſtraight line intercepted between the perpendicular let fall upon it from the oppoſite angle, and the acute angle.

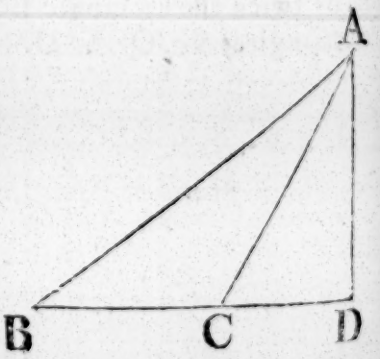
Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the ſides containing it, let fall
a. 12. 1. the perpendicular [a] AD from the oppoſite angle: The ſquare of AC, oppoſite to the angle B, is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.

Firſt, Let AD fall within the triangle ABC; and becauſe the ſtraight line CB is divided into two parts in the point D, the



b. 7. 2. ſquares of CB, BD are equal [b] to twice the rectangle contained by CB, BD, and the ſquare of DC: To each of theſe equals add the ſquare of AD; therefore the ſquares of CB, BD, DA are equal to twice the rectangle CB, BD, and the ſquares of AD, DC: But the ſquare of AB is equal
c. 47. 1. [c] to the ſquares of BD, DA, becauſe the angle BDA is a right angle; and the ſquare of AC is equal to the ſquares of AD, DC: Therefore the ſquares of CB, BA, are equal to the ſquare of AC, and twice the rectangle CB, BD; that is, the ſquare of AC alone is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.

Secondly, Let AD fall without the triangle ABC: Then, becauſe the angle at D is a right angle, the angle ACB is greater

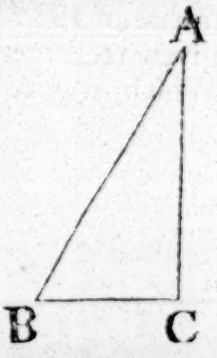


d 16. 1. [d] than a right angle; and therefore the ſquare of AB is equal
e. 12. 2. [e] to the ſquares of AC, CB, and twice the rectangle BC, CD: To theſe equals add the ſquare of BC, and the

ſquares

uares of AB, BC are equal to the ſquare of AC, and twice Book II.
e ſquare of BC, and twice the rectangle BC, CD: But be-
uſe BD is divided into two parts in C, the rectangle DB, BC
equal [f] to the rectangle BC, CD and the ſquare of BC: And f. 3.
e doubles of theſe are equal: Therefore the ſquares of AB,
C are equal to the ſquare of AC, and twice the rectangle
B, BC: Therefore the ſquare of AC alone is leſs than the
uares of AB, BC by twice the rectangle DB, BC.

Laſtly, Let the ſide AC be perpendicular to
C; then is BC the ſtraight line between the
rpendicular and the acute angle at B; and it
manifeſt that the ſquares of AB, BC are e-
al [c] to the ſquare of AC, and twice the c. 47. 1.
uare of BC: Therefore, in every triangle, &c.
E. D.

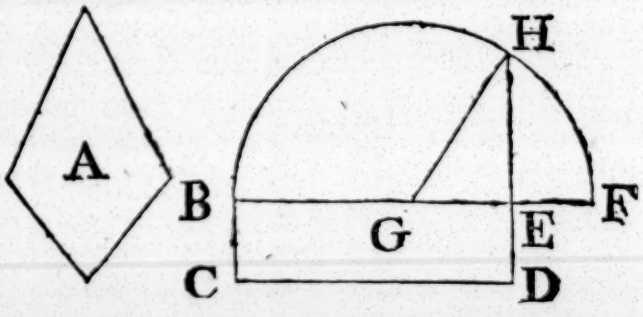


PROP. XIV. THEOR.

TO deſcribe a ſquare that ſhall be equal to a given rectilineal figure. See N.

Let A be the given rectilineal figure; it is required to de-
ibe a ſquare that ſhall be equal to A.

Deſcribe [a] the rectangular parallelogram BCDE equal to the a. 45. 1.
ctilineal figure A. If then the ſides of it BE, ED are equal
one another, it
a ſquare, and
hat was required
now done: But
they are not e-
ual, produce one
them BE to F,
nd make EF equal



ED, and biſect BF in G; and from the center G, at the di-
ance GB, or GF, deſcribe the ſemicircle BHF, and produce
E to H, and join GH: Therefore, becauſe the ſtraight line BF
divided into two equal parts in the point G, and into two
nequal at E, the rectangle BE, EF, together with the ſquare
EG, is equal [b] to the ſquare of GF: But GF is equal to b. 5. 2.
H; therefore the rectangle BE, EF, together with the ſquare
EG, is equal to the ſquare of GH: But the ſquares of HE,

EG

Book. II. EG are equal [c] to the ſquare of GH: Therefore the rectangle BE, EF, together with the ſquare of EG, is equal to the ſquares of HE, EG: Take away the ſquare of EG, which is common to both; and the remaining rectangle BE, EF is equal to the ſquare of EH: But the rectangle contained by BE, EF is the parallelogram BD, becauſe EF is equal to ED; therefore BD is equal to the ſquare of EH: But BD is equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the ſquare of EH: Wherefore a ſquare has been made equal to the given rectilineal figure A, viz. the ſquare deſcribed upon EH. Which was to be done.

c. 47. 1.

THE ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

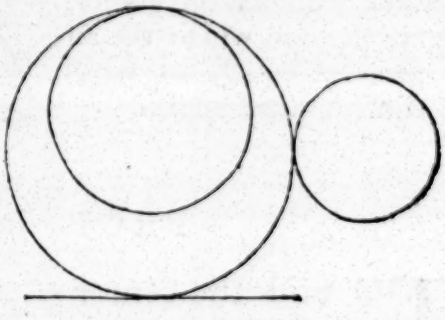
I.

EQUAL circles are thoſe of which the diameters are equal, or from the centers of which the ſtraight lines to the circumferences are equal.

'This is not a definition but a theorem, the truth of which is evident; for if the circles be applied to one another, ſo that their centers coincide, the circles muſt likewiſe coincide, ſince the ſtraight lines from their centers are equal.'

II.

A ſtraight line is ſaid to touch a circle, when it meets the circle, and being produced does not cut it.

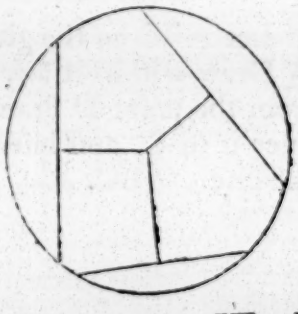


III.

Circles are ſaid to touch one another, which meet but do not cut one another.

IV.

Straight lines are ſaid to be equally diſtant from the center of a circle, when the perpendiculars drawn to them from the center are equal.



V.

And the ſtraight line on which the greater perpendicular falls, is ſaid to be farther from the center.

Book III.

VI.

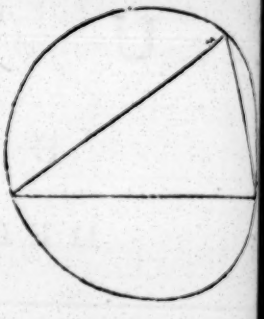
A ſegment of a circle is the figure contained by a ſtraight line and the circumference it cuts off.

VII.

"The angle of a ſegment is that which is contained by the "ſtraight line and the circumference."

VIII.

An angle in a ſegment is the angle contained by two ſtraight lines drawn from any point in the circumference of the ſegment, to the extremities of the ſtraight line which is the baſe of the ſegment.



IX.

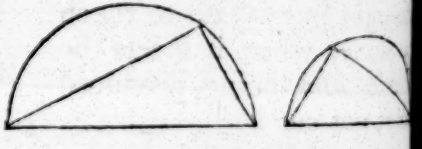
And an angle is ſaid to inſiſt or ſtand upon the circumference intercepted between the ſtraight lines that contain the angle.

X.

The ſector of a circle is the figure contained by two ſtraight lines drawn from the center, and the circumference between them.

XI.

Similar ſegments of a circle, are thoſe in which the angles are equal, or which contain equal angles.



PROP. I. PROB.

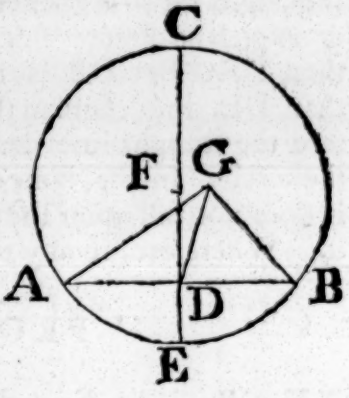
TO find the center of a given circle.

Let ABC be the given circle; it is required to find its center.

a. 10. 1. Draw within it any ſtraight line AB, and biſect [a] it in D;

b. 11. 1. from the point D draw [b] DC at right angles to AB, and produce it to E, and biſect CE in F: The point F is the center of the circle ABC.

For, if it be not, let, if poſſible, G be the center, and join A, GD, GB: Then, becauſe DA is equal to DB, and DG mmon to the two triangles ADG, G, the two ſides AD, DG are e- l to the two BD, DG, each to h; and the baſe GA is equal to baſe GB, becauſe they are drawn m the center G †: Therefore the le ADG is equal [c] to the angle B: But when a ſtraight line ſtand- upon another ſtraight line makes adjacent angles equal to one ano- r, each of the angles is a right an- [d]: Therefore the angle GDB is a t angle: But FDB is likewiſe a right angle; wherefore the le FDB is equal to the angle GDB, the greater to the leſs, ich is impoſſible: Therefore G is not the center of the circle C: In the ſame manner it can be ſhewn, that no other nt but F is the center; that is, F is the center of the circle C; Which was to be found.



c. 8. 1.

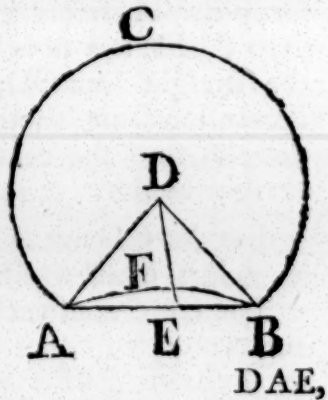
d. 10. def. 1.

Cor. From this it is manifeſt, that if in a circle a ſtraight biſect another at right angles, the center of the circle is in line which biſects the other.

PROP. II. THEOR.

F any two points be taken in the circumference of a circle, the ſtraight line which joins them ſhall l within the circle.

Let ABC be a circle, and A, B any two points in the cir- mference; the ſtraight line drawn m A to B ſhall fall within the circle. For, if it do not, let it fall, if poſ- e, without, as AEB; find [a] D the ter of the circle ABC, and join AD, B, and produce DF any ſtraight line eting the circumference AB, to E: en becauſe DA is equal to DB, the le DAB is equal [b] to the angle DBA; d becauſe AE, a ſide of the triangle DAE,



a 1. 3.

b. 5. 1.

† N B. Whenever the expreſſion "ſtraight lines from the center" or "drawn om the center" occurs, it is to be underſtood that they are drawn to the circum- nce,

Book III.

DAE, is produced to B, the angle DEB is greater [c] than
c. 16. 3. angle DAE; but DAE is equal to the angle DBE; there
the angle DEB is greater than the angle DBE: But to the gr
d. 19. 1. er angle the greater ſide is oppoſite [d]; DB is therefore gre
than DE: But DB is equal to DF; wherefore DF is gre
than DE, the leſs than the greater, which is impoſſible: Th
fore the ſtraight line drawn from A to B does not fall with
the circle. In the ſame manner, it may be demonſtrated
it does not fall upon the circumference; it falls therefore wi
it. Wherefore, if any two points, &c. Q. E. D.

PROP. III. THEOR.

IF a ſtraight line drawn thro' the center of a cir
biſect a ſtraight line in it which does not p
thro' the center, it ſhall cut it at right angles: A
if it cuts it at right angles, it ſhall biſect it.

Let ABC be a circle; and let CD, a ſtraight line dr
thro' the center biſect any ſtraight line AB, which does
paſs thro' the center, in the point F: It cuts it alſo at r
angles.

a. 1. 3. Take [a] E the center of the circle, and join EA, EB: Th
becauſe AF is equal to FB, and FE common to the two
angles AFE, BFE, there are two ſides in the one equal to
ſides in the other, and the baſe EA is
equal to the baſe EB; therefore the
b. 8. 1. angle AFE is equal [b] to the angle BFE:
But when a ſtraight line ſtanding upon
another makes the adjacent angles equal
to one another, each of them is a right
c. 10. def. 1. [c] angle: Therefore each of the angles
AFE, BFE is a right angle; wherefore
the ſtraight line CD, drawn thro' the
center biſecting another AB that does
not paſs thro' the center, cuts the ſame
at right angles.

But let CD cut AB at right angles; CD alſo biſects it,
is, AF is equal to FB.

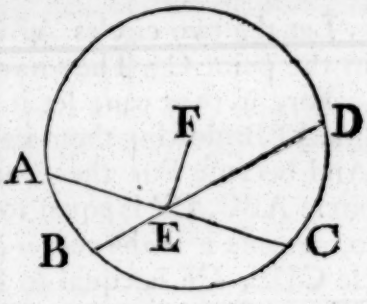
The ſame conſtruction being made, becauſe EA, EB fr
d. 5. 1. the center are equal to one another, the angle EAF is equ
to the angle EBF; and the right angle AFE is equal to
right angle BFE: Therefore, in the two triangles EAF, E
th

ere are two angles in one equal to two angles in the other, Book. III.
d the ſide EF, which is oppoſite to one of the equal angles
each, is common to both; therefore the other ſides are e-
al [c]; AF therefore is equal to FB. Wherefore, if a ſtraight c. 26. 1.
e, &c. Q. E. D.

PROP. IV. THEOR.

F in a circle two ſtraight lines cut one another
which do not both paſs thro' the center, they do
t biſect each the other.

Let ABCD be a circle, and AC, BD two ſtraight lines in
which cut one another in the point E, and do not both paſs
ro' the center. AC, BD do not biſect one another.
For, if it is poſſible, let AE be equal to EC, and BE to ED:
one of the lines paſs thro' the center, it is plain that it can-
t be biſected by the other which
es not paſs thro' the center: But, if
ither of them paſs thro' the center,
ke [a] F the center of the circle, and
n EF: And becauſe FE, a ſtraight
e thro' the center, biſects another
C which does not paſs thro' the
nter, it ſhall cut it at right [b] angles;
herefore FEA is a right angle: A-
in, becauſe the ſtraight line FE bi-
ts the ſtraight line BD which does not paſs thro' the center,
ſhall cut it at right [b] angles; wherefore FEB is a right angle:
nd FEA was ſhewn to be a right angle; therefore FEA is
ual to the angle FEB, the leſs to the greater, which is im-
ſible: Therefore AC, BD do not biſect one another. Where-
e, if in a circle, &c. Q. E. D.



a. 1. 3.
b. 3. 3.

PROP. V. THEOR.

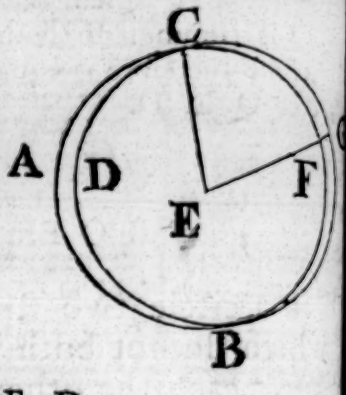
F two circles cut one another, they ſhall not have
the ſame center.

Let the two circles ABC, CDG cut one another in the points
C; they have not the ſame center.

For,

Book. III.

For, if it be poſſible, let E be their center: Join EC, a draw any ſtraight line EFG meeting them in F and G: And becauſe E is the center of the circle ABC, CE is equal to EF: Again, becauſe E is the center of the circle CDG, CE is equal to EG: But CE was ſhewn to be equal to EF; therefore EF is equal to EG, the leſs to the greater, which is impoſſible: Therefore E is not the center of the circles ABC, CDG. Wherefore, if two circles, &c. Q. E. D.

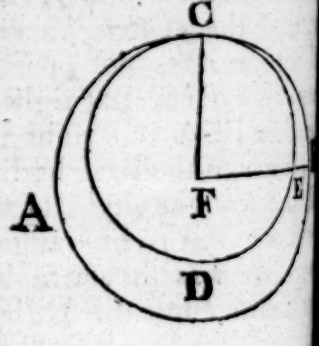


PROP. VI. THEOR.

IF two circles touch one another internally, th ſhall not have the ſame center.

Let the two circles ABC, CDE touch one another intern in the point C: They have not the ſame center.

For, if they can, let it be F; join FC, and draw any ſtrai line FEB meeting them in E and B: And becauſe F is the center of the circle ABC, CF is equal to FB: Alſo, becauſe F is the center of the circle CDE, CF is equal to FE: And CF was ſhewn equal to FB; therefore FE is equal to FB, the leſs to the greater, which is impoſſible: Wherefore F is not the center of the circles ABC, CDE. Therefore, if two circles, Q. E. D.



PRO

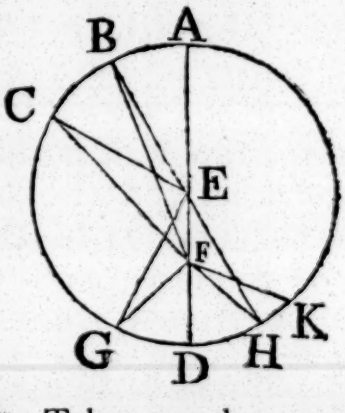
PROP. VII. THEOR.

IF any point be taken in the diameter of a circle, which is not the center, of all the ſtraight lines which can be drawn from it to the circumference, the greateſt is that in which the center is, and the other part of that diameter is the leaſt; and of any others, that which is nearer to the line which paſſes through the center is always greater than one more remote: And from the ſame point there can be drawn only two ſtraight lines that are equal to one another, one upon each ſide of the ſhorteſt line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the center: Let the center be E; of all the ſtraight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greateſt, and FD, the other part of the diameter AD, is the leaſt; and of the others, FB is greater than FC, and FC than FG.

Join BE, CE, GE; and becauſe two ſides of a triangle are greater [a] than the third, BE, EF are greater than BF; but AE is equal to EB, therefore AE, EF, that is AF, is greater than BF: Again, becauſe BE is equal to CE, and FE common to the triangles BEF, CEF, the two ſides BE, EF are equal to the two CE, EF; but the angle BEF is greater than the angle CEF, therefore the baſe BF is greater [b] than the baſe FC: For the ſame reaſon, CF is greater than GF: Again, becauſe GF, FE are greater than EG, and EG is equal to ED; GF, FE are greater than ED: Take away the common part FE, and the remainder GF is greater than the remainder FD: Therefore FA is the greateſt, and FD the leaſt of all the ſtraight lines from F to the circumference; and BF is greater than CF, and CF than GF.

a. 20. 1.



b. 24. 1.

Alſo there can be drawn only two equal ſtraight lines from the point F to the circumference, one upon each ſide of the

Book III. ſhorteſt line FD: At the point E in the ſtraight line EF, mak[e]
[c] the angle FEH equal to the angle GEF, and join FH: The
c. 23. 1. becauſe GE is equal to EH, and EF common to the two trian
gles GEF, HEF; the two ſides GE, EF are equal to the tw
HE, EF; and the angle GEF is equal to the angle HEF, there
d. 4. 1. fore the baſe FG is equal [d] to the baſe FH: But beſides FH n
other ſtraight line can be drawn from F to the circumf
rence equal to FG: For, if there can, let it be FK; and becau
FK is equal to FG, and FG to FH, FK is equal to FH, that i
a line nearer to that which paſſes through the center, is equal t
one which is more remote; which is impoſſible. Therefore, i
any point be taken, &c. Q. E. D.

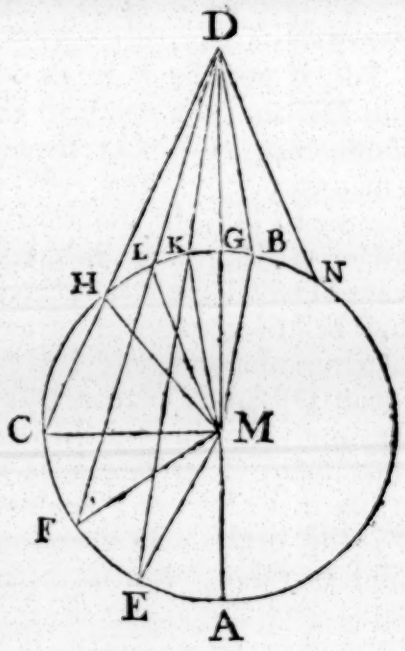
PROP. VIII. THEOR.

IF any point be taken without a circle, and ſtraight lines be drawn from it to the circumference whereof one paſſes through the center; of thoſe which fall upon the concave circumference, the greateſt is that which paſſes through the center; and of the reſt, that which is nearer to that through the center is always greater then the more remote. But of thoſe which fall upon the convex circumference, the leaſt is that between the point without the circle, and the diameter; and of the reſt, that which is nearer to the leaſt is always leſs than the more remote: And only two equal ſtraight lines can be drawn from the point unto the circumference, one upon each ſide of the leaſt.

Let ABC be a circle, and D any point without it, from which let the ſtraight lines DA, DE, DF, DC be drawn to the circumference, whereof DA paſſes through the center. Of thoſe which fall upon the concave part of the circumference AEFC the greateſt is AD which paſſes through the center; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DC: But of thoſe which fall upon the convex circumference HLKG, the leaſt is DG between the point

oint D and the diameter AG; and the nearer to it is always eſs than the more remote, viz. DK than DL, and DL than DH.

Take [a] M the center of the circle ABC, and join ME, MF, MC, MK, ML, MH: And becauſe AM is equal to ME, add MD to each, therefore AD is equal to EM, MD; but EM, MD re greater [b] than ED, therefore alſo AD is greater than ED: Again, becauſe ME is equal to MF, and MD common to the riangles EMD, FMD; EM, MD re equal to FM, MD; but the ngle EMD is greater than the ngle FMD; therefore the baſe D is greater [c] than the baſe FD: n like manner it may be ſhewn hat FD is greater than CD: Therefore DA is the greateſt; and DE greater than DF, and DF than DC: And becauſe MK, KD are reater [b] than MD, and MK is e- ual to MG, the remainder KD greater [d] than the remainder GD, hat is, GD is leſs than KD: And ecauſe MK, DK are drawn to the oint K within the triangle MLD rom M, D the extremities of its de MD; MK, KD are leſs [e] than ML, LD, whereof MK is equal o ML, therefore the remainder DK is leſs than the remainder DL: In like manner it may be ſhewn, that DL is leſs than DH: Therefore DG is the leaſt, and DK leſs than DL, and DL han DH: Alſo there can be drawn only two equal ſtraight nes from the point D to the circumference, one upon each de of the leaſt: At the point M, in the ſtraight line MD, make he angle DMB equal to the angle DMK, and join DB: And ecauſe MK is equal to MB, and MD common to the triangles KMD, BMD, the two ſides KM, MD are equal to the two BM, MD; and the angle KMD is equal to the angle BMD, there- ore the baſe DK is equal [f] to the baſe DB: But beſides DB here can be no ſtraight line drawn from D to the circumfe- ence equal to DK: For, if there can, let it be DN; and becauſe DK is equal to DN, and alſo to DB, therefore DB is equal to DN, that is, the nearer to the leaſt equal to the more remote, which is impoſſible. If therefore, any point, &c. Q. E. D.



[a] 1. 3.

[b] 20. 1.

[c] 24. 1.

[d] 4. Ax.

[e] 21. 1.

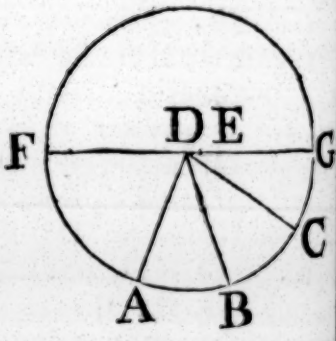
[f] 4. 1.

Book. III.

PROP. IX. THEOR.

IF a point be taken within a circle, from which there fall more than two equal ſtraight lines to the circumference, that point is the center of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal ſtraight lines, viz. DA, DB, DC, the point D is the center of the circle.

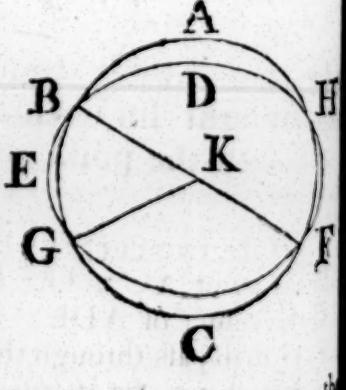


For, if not, let E be the center, join DE and produce it to the circumference in F, G; then FG is a diameter of the circle ABC: And becauſe in FG, the diameter of the circle ABC, there is taken the point D which is not the center, DG ſhall be the greateſt line from it to the circumference, and DC greater [a] than DB, and DB than DA: But they are likewiſe equal, which is impoſſible: Therefore E is not the center of the circle ABC. In like manner it may be demonſtrated, that no other point but D is the center; D therefore is the center. Wherefore, if a point be taken, &c. Q. E. D.

[a] 7. 3.

PROP. X. THEOR.

ONE circumference of a circle cannot cut another in more than two points.



If it be poſſible, let the circumference ABC cut the circumference DEF in more than two points, viz. in B, G, F; take the center K of the circle ABC, and join KB, KG, KF: And becauſe within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal ſtraight lines KB, KG, KF, the point K is [a]

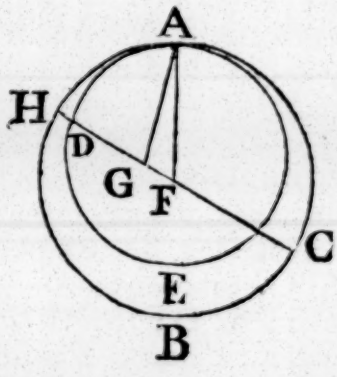
[a] 9. 3.

the center of the circle DEF: But K is alſo the center of the circle ABC; therefore the ſame point is the center of two circles that cut one another, which is impoſſible [b]. Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D. Book III. [b] 5. 3.

PROP. XI. THEOR.

IF two circles touch each other internally, the ſtraight line which joins their centers being produced ſhall paſs through the point of contact.

Let the two circles ABC, ADE touch each other internally in the point A, and let F be the center of the circle ABC, and G the center of the circle ADE: The ſtraight line which joins the centers F, G, being produced, paſſes through the point A.



For, if not, let it fall otherwiſe, if poſſible, as FGDH, and join AF, AG: And becauſe AG, GF are greater [a] than FA, that is, than FH, for FA is equal to FH, both being from the ſame center; take away the common part FG, therefore the remainder AG is greater than the remainder GH: But AG is equal to GD, therefore GD is greater than GH, the leſs than the greater, which is impoſſible. Therefore the ſtraight line which joins the points F, G cannot fall otherwiſe than upon the point A, that is, it muſt paſs through it. Therefore, if two circles, &c. Q. E. D. [a] 20. 1.

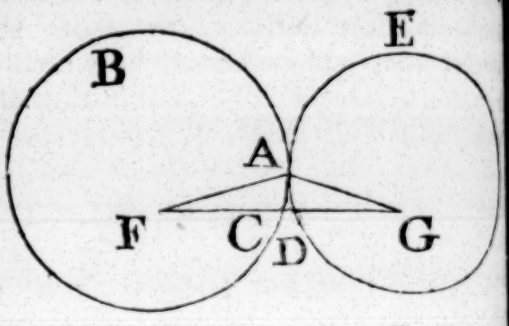
PROP. XII. THEOR.

IF two circles touch each other externally, the ſtraight line which joins their centers ſhall paſs through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the center of the circle ABC, and G the center of ADE: The ſtraight line which joins the points F, G ſhall paſs through the point of contact A.

For, if not, let it paſs otherwiſe, if poſſible, as FCDG, and

Book III. join FA, AG: And becauſe F is the center of the circle ABC, AF is equal to FC: Alſo becauſe G is the center of the circle ADE, AG is equal to GD: Therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG: But it is alſo leſs [a]; which is impoſſible: Therefore the ſtraight line which joins the points F, G ſhall not paſs otherwiſe than through the point of contact A, that is, it muſt paſs through it. Therefore, if two circles, &c. Q. E. D.



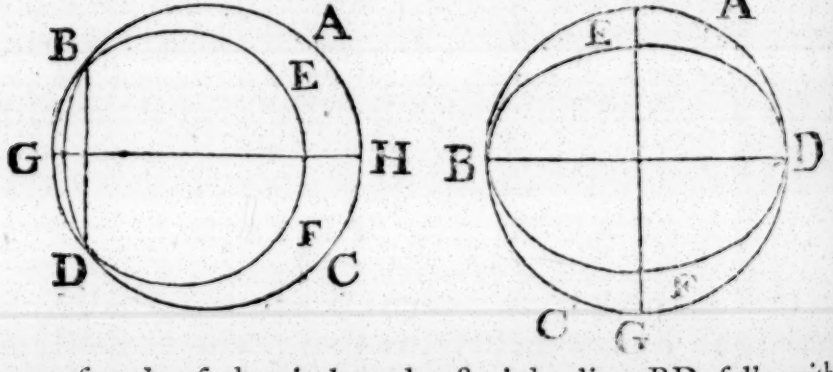
[a] 20. 1.

PROP. XIII. THEOR.

See N. ONE circle cannot touch another in more points than one, whether it touches it on the inſide or outſide.

For, if it be poſſible, let the circle EBF touch the circle ABC in more points than one, and firſt on the inſide, in the points B, D; join BD, and draw [a] GH biſecting BD at right angles: Therefore, becauſe the points B, D are in the circumfe-

[a] 10. 11. 1.

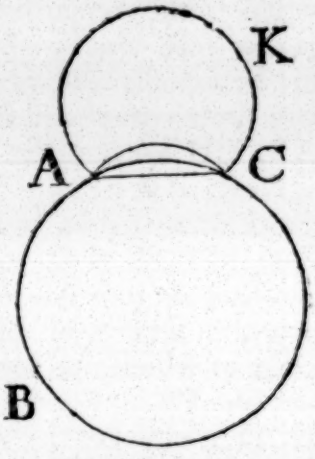


rence of each of the circles, the ſtraight line BD falls within each of them: And their centers are [c] in the ſtraight line GH which biſects BD at right angles; therefore GH paſſes through the point of contact [d]; but it does not paſs through it, becauſe the points B, D are without the ſtraight line GH, which is abſurd: Therefore one circle cannot touch another on the inſide in more points than one.

[b] 2. 3.
[c] Cor. 1. 3.
[d] 11. 3.

Nor can two circles touch one another on the outſide in more

more than one point: For, if it be poſſible, let the circle ACK touch the circle ABC in the points A, C, and join AC: Therefore, becauſe the two points A, C are in the circumference of the circle ACK, the ſtraight line AC which joins them ſhall fall within [b] the circle ACK: And the circle ACK is without the circle ABC, and therefore the ſtraight line AC is without this laſt circle; but, becauſe the points A, C are in the circumference of the circle ABC, the ſtraight line AC muſt be within [b] the ſame circle, which is abſurd: Therefore one circle cannot touch another on the outſide in more than one point: And it has been ſhewn, that they cannot touch on the inſide in more points than one: Therefore, one circle, &c. Q. E D.



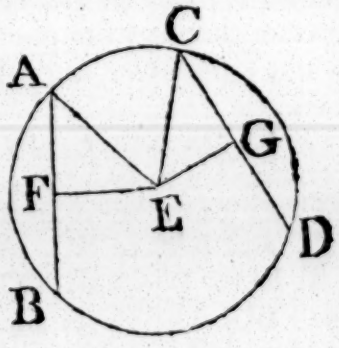
[b] 2. 3.

PROP. XIV. THEOR.

EQUAL ſtraight lines in a circle are equally diſtant from the center; and thoſe which are equally diſtant from the center, are equal to one another.

Let the ſtraight lines AB, CD, in the circle ABDC, be equal to one another; they are equally diſtant from the center.

Take E the center of the circle ABDC, and from it draw EF, EG perpendiculars to AB, CD: Then, becauſe the ſtraight line EF paſſing through the center cuts the ſtraight line AB, which does not paſs through the center, at right angles, it alſo biſects [a] it: Wherefore AF is equal to FB, and AB double of AF. For the ſame reaſon CD is double of CG: And AB is equal to CD, therefore AF is equal to CG: And becauſe AE is equal to EC, the ſquare of AE is equal to the ſquare of EC: But the ſquares of AF, FE are equal [b] to the ſquare of AE, becauſe the angle AFE is a right angle; and for the like reaſon the ſquares of EG, GC are equal to the ſquare of EC: Therefore the ſquares of AF, FE are equal to the ſquares of CG, GE, of which



[a] 3. 3.

[b] 47. 1.

Book III. which the ſquare of AF is equal to the ſquare of CG, becauſe AF is equal to CG; therefore the remaining ſquare of FE is equal to the remaining ſquare of EG, and the ſtraight line FE is therefore equal to EG: But ſtraight lines in a circle are ſaid to be equally diſtant from the center, when the perpendiculars drawn to them from the center are equal [c]: Therefore AB, CD are equally diſtant from the center.

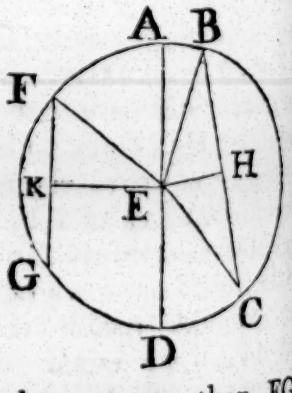
[c] 4. Def. 3.

Next, if the ſtraight lines AB, CD be equally diſtant from the center, that is, if FE be equal to EG; AB is equal to CD. For, the ſame conſtruction being made, it may, as before, be demonſtrated, that AB is double of AF and CD double of CG, and that the ſquares of EF, FA are equal to the ſquares of EG, GC; of which the ſquare of FE is equal to the ſquare of EG, becauſe FE is equal to EG; therefore the remaining ſquare of AF is equal to the remaining ſquare of CG; and the ſtraight line AF is therefore equal to CG: And AB is double of AF, and CD double of CG; wherefore AB is equal to CD. Therefore equal ſtraight lines, &c. Q. E. D.

PROP. XV. THEOR.

See N. THE diameter is the greateſt ſtraight line in a circle; and of all others, that which is nearer to the center is always greater than one more remote; and the greater is nearer to the center than the leſs.

Let ABCD be a circle, of which the diameter is AD, and center E; and let BC be nearer to the center than FG; AD is greater than any ſtraight line BC which is not a diameter, and BC greater than FG.



From the center draw EH, EK perpendiculars to BC, FG, and join EB, EC, EF; and becauſe AE is equal to EB, and ED to EC, AD is equal to EB, EC: But EB, EC, are greater [a] than BC, wherefore, alſo AD is greater than BC.

[a] 20. I.

And becauſe BC is nearer to the center than FG, EH

H is leſs [b] than EK: But, as was demonſtrated in the preceeding, BC is double of BH, and FG double of FK, and the squares of EH, HB are equal to the ſquares of EK, KF, of which the ſquare of EH is leſs than the ſquare of EK, becauſe EH is leſs than EK; therefore the ſquare of BH is greater than the ſquare of FK, and the ſtraight line BH greater than FK; and therefore BC is greater than FG. Book III. [b] 5. def. 3.

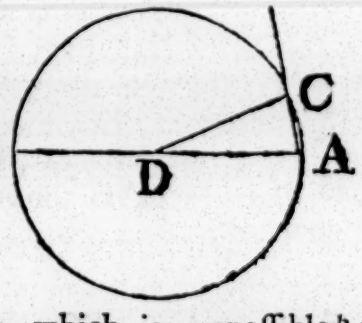
Next, Let BC be greater than FG; BC is nearer to the center than FG, that is, the ſame conſtruction being made, EH is leſs than EK: Becauſe BC is greater than FG, BH likewiſe is greater than FK: And the ſquares of BH, HE are equal to the ſquares of FK, KE, of which the ſquare of BH is greater than the ſquare of FK, becauſe BH is greater than FK; therefore the ſquare of EH is leſs than the ſquare of EK, and the ſtraight line EH leſs than EK. Wherefore the diameter, &c. Q. E. D.

PROP. XVI. THEOR.

THE ſtraight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no ſtraight line can be drawn between that ſtraight line and the circumference from the extremity, ſo as not to cut the circle; or, which is the ſame thing, no ſtraight line can make ſo great an acute angle with the diameter at its extremity, or ſo ſmall an angle with the ſtraight line which is at right angles to it, as not to cut the circle. See N.

Let ABC be a circle the center of which is D, and the diameter AB; the ſtraight line drawn at right angles to AB from its extremity A, ſhall fall without the circle.

For, if it does not, let it fall, if poſſible, within the circle as AC, and draw DC to the point C where it meets the circumference: And becauſe DA is equal to DC, the angle DAC is equal [a] to the angle ACD; but DAC is a right angle, therefore ACD is a right angle, and the angles DAC, ACD are therefore equal to two right angles; which is impoſſible [b]: Therefore

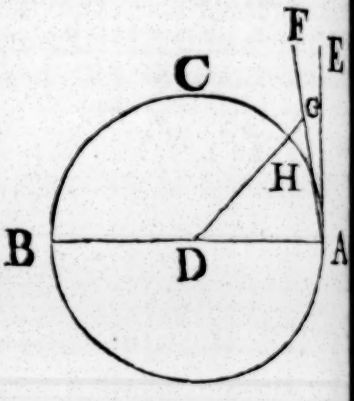


[a] 5. 1.

[b] 17. 1.

Book III. Therefore the ſtraight line drawn from A at right angles to BA does not fall within the circle: In the ſame manner it may be demonſtrated that it does not fall upon the circumference; therefore it muſt fall without the circle, as AE.

And between the ſtraight line AE and the circumference no ſtraight line can be drawn from the point A which does not cut the circle: For, if poſſible, let FA be between them, and [c] 12. 1. from the point D draw [c] DG perpendicular to FA, and let it meet the circumference in H: And becauſe AGD is a right [d] 19. 1. angle, and DAG leſs [b] than a right angle: DA is greater [d] than DG: But DA is equal to DH; therefore DH is greater than DG, the leſs than the greater, which is impoſſible: Therefore no ſtraight line can be drawn from the point A between AE and the circumference, which does not cut the circle, or, which amounts to the ſame thing, however great an acute angle a ſtraight line makes with the diameter at the point A, or however ſmall an angle it makes with AE, the circumference paſſes between that ſtraight line and the perpendicular AE. 'And this is all that is to be underſtood 'when, in the Greek text and tranſlations from it, the angle of 'the ſemicircle is ſaid to be greater than any acute rectilineal 'angle, and the remaining angle leſs than any rectilineal angle.'



Cor. From this it is manifeſt that the ſtraight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, becauſe, if it did meet the circle in two, it would fall [e] 2. 3. within it [e]. 'Alſo it is evident that there can be but one 'ſtraight line which touches the circle in the ſame point.'

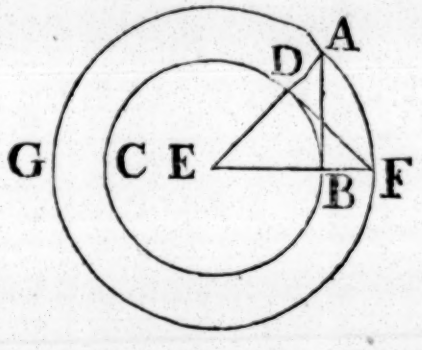
PROP. XVII. PROB.

TO draw a ſtraight line from a given point, either without or in the circumference, which ſhall touch a given circle.

Firſt, Let A be a given point without the given circle BCD;

is required to draw a ſtraight line from which A ſhall touch the circle. Book III.

Find [a] the center E of the circle, and join AE; and from the center E, at the diſtance EA, deſcribe the circle AFG; from the point D draw [b] DF at right angles to EA, and join DF, AB: AB touches the circle BCD. [a] 1. 3. [b] 11. 1.



Becauſe E is the center of the circles BCD, AFG, AE is equal to EF: And ED to EB; therefore the two ſides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the baſe DF is equal to the baſe AB, and the triangle FED to the triangle AEB, and the other angles to the other angles [c]: Therefore the angle EBA is equal to the angle EDF: But EDF is a right angle, wherefore EBA is a right angle: And EB is drawn from the center; but a ſtraight line drawn from the extremity of a diameter, at right angles to it, touches the circle [d]: Therefore AB touches the circle; and it is drawn from the given point A. which was to be done. [c] 4. 1. [d] Cor. 16. 3.

But if the given point be in the circumference of the circle, as the point D, draw DE to the center E, and DF at right angles to DE; DF touches the circle [d].

PROP. XVIII. THEOR.

IF a ſtraight line touches a circle, the ſtraight line drawn from the center to the point of contact, ſhall be perpendicular to the line touching the circle.

Let the ſtraight line DE touch the circle ABC in the point B, take the center F, and draw the ſtraight line FC; FC is perpendicular to DE.

For, if it be not, from the point F draw FBG perpendicular to DE; and becauſe FGC is a right angle, GCF is [b] an acute angle; and to the greater angle the greateſt [c] ſide is oppoſite: [b] 17. 1. [c] 19. 1.

Therefore

Book III.

Therefore FC is greater than FG; but FC is equal to FB; therefore FB is greater than FG, the leſs than the greater, which is impoſſible: Wherefore FG is not perpendicular to DE: In the ſame manner it may be ſhewn, that no other is perpendicular to it beſides FC, that is, FC is perpendicular to DE. Therefore, if a ſtraight line, &c. Q. E. D.

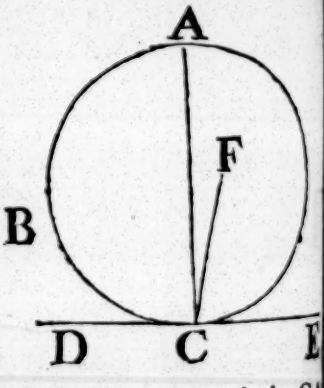
PROP. XIX. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn at right angles to the touching line, the center of the circle ſhall be in that line.

Let the ſtraight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the center of the circle is in CA.

For, if not, let F be the center, if poſſible, and join CF. Becauſe DE touches the circle ABC, and FC is drawn from the center to the point of contact, FC is perpendicular [a] to DE; therefore FCE is a right angle: But ACE is alſo a right angle; therefore the angle FCE is equal to the angle ACE, the leſs to the greater, which is impoſſible: Wherefore F is not the center of the circle ABC: In the ſame manner it may be ſhewn that no other point which is not in CA, is the center; that is, the center is in CA. Therefore, if a ſtraight line, &c. Q. E. D.

[a] 18. 3.

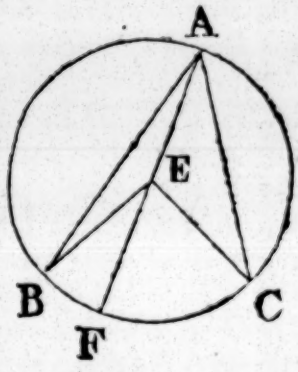


PROP. XX. THEOR.

See N. THE angle at the center of a circle is double of the angle at the circumference, upon the ſame baſe, that is, upon the ſame part of the circumference

L

Let ABC be a circle, and BEC an angle at the center, and Book III.
AC an angle at the circumference, which have the ſame cir-
mference BC for their baſe; the angle
EC is double of the angle BAC.

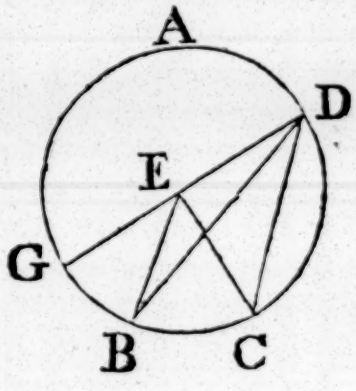


[a] 5. 1.

[b] 32. 1.

Firſt, Let E the center of the circle be
ithin the angle BAC, and join AE, and
oduce it to F: Becauſe EA is equal
EB, the angle EAB is equal [a] to
e angle EBA; therefore the angles
AB, EBA are double of the angle EAB;
t the angle BEF is equal [b] to the angles
AB, EBA; therefore alſo the angle
EF is double of the angle EAB: For the
me reaſon, the angle FEC is double of the angle EAC: There-
re the whole angle BEC is double of the whole angle BAC.

Again, Let E the center of the
rcle be without the angle BDC, and
in DE and produce it to G. It
ay be demonſtrated, as in the firſt
ſe, that the angle GEC is double
the angle GDC, and that GEB a
rt of the firſt is double of GDB a
art of the other; therefore the re-
aining angle BEC is double of the
maining angle BDC. Therefore
e angle at the center, &c. Q. E. D.

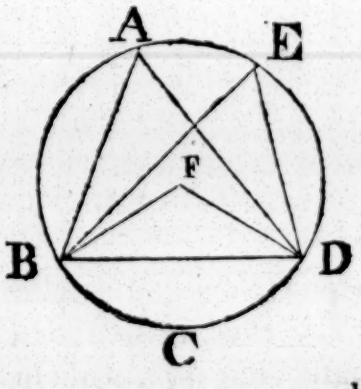


PROP. XXI. THEOR.

THE angles in the ſame ſegment of a circle are e- See N.
qual to one another.

Let ABCD be a circle, and BAD,
ED angles in the ſame ſegment
AED: the angles BAD, BED are
qual to one another.

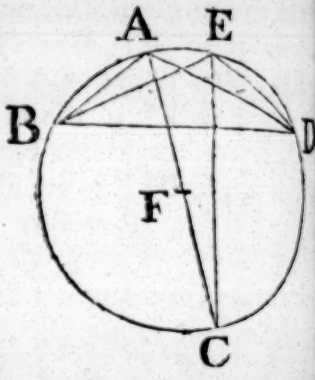
Take F the center of the circle
BCD: And, firſt, let the ſegment
AED be greater than a ſemicircle,
nd join BF, FD: And becauſe the
ngle BFD is at the center, and the
ngle BAD at the circumference,
nd that they have the ſame part of



the

Book III. the circumference, viz BCD, for their baſe; therefore the a-
gle BFD is double [a] of the angle BAD: For the ſame reaſo
[a] 20. 3. the angle BFD is double of the angle BED: Therefore the a-
gle BAD is equal to the angle BED.

But, if the ſegment BAED be not greater than ſemicirc
let BAD, BED be angles in it; theſe alſo are equal to one another: Draw AF to the center, and produce it to C, and join CE: Therefore the ſegment BADC is greater than a ſemicircle; and the angles in it BAC, BEC are equal, by the firſt caſe: For the ſame reaſon, becauſe CBED is greater than a ſemicircle, the angles CAD, CED are equal: Therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the ſame ſegme
&c. Q. E. D.



PROP. XXII. THEOR.

THE oppoſite angles of any quadrilateral figur
deſcribed in a circle, are together equal to tw
right angles.

Let ABCD be a quadrilateral figure in the circle ABCD
any two of its oppoſite angles are together equal to two rig
angles.

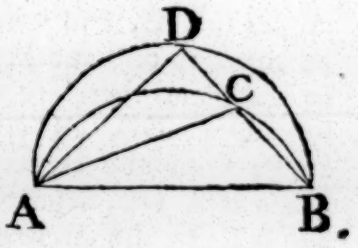
Join AC, BD; and becauſe the three angles of every t
[a] 32. 1. angle are equal [a] to two right angles, the three angles of th
triangle CAB, viz. the angles CAB, ABC, BCA are equal t
two right angles: But the angle CAB
[b] 21. 3. is equal [b] to the angle CDB, becauſe they are in the ſame ſegment BADC; and the angle ACB is equal to the angle ADB, becauſe they are in the ſame ſegment ADCB: Therefore the whole angle ADC is equal to the angles CAB, ACB: To each of theſe equals add the angle ABC, therefore the angles ABC, CAB, BCA are equal to the angles ABC, ADC: But ABC, CAB, BCA a
equal to two right angles; therefore alſo the angles ABC, AD
are equal to two right angles: In the ſame manner the angl
BA

AD, DCB may be ſhewn to be equal to two right angles. Therefore, the oppoſite angles, &c. Q. E. D. Book II

PROP. XXIII. THEOR.

UPON the ſame ſtraight line, and upon the ſame ſide of it, there cannot be two ſimilar ſegments of circles, not coinciding with one another. See N.

If it be poſſible, let the two ſimilar ſegments of circles, viz. ACB, ADB, be upon the ſame ſide of the ſame ſtraight line AB, not coinciding with one another: Then, becauſe the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point [a]: One of the ſegments muſt therefore fall within the other; let ACB fall within ADB, and draw the ſtraight line BCD, and join CA, DA: And becauſe the ſegment ACB is ſimilar to the ſegment ADB, and that ſimilar ſegments of circles contain [b] equal angles; the angle ACB is equal to the angle ADB, the exterior to the interior, which is impoſſible [c]. Therefore, there cannot be two ſimilar ſegments of a circle upon the ſame ſide of the ſame line, which do not coincide. Q. E. D.



[a] 10. 3.

[b] 11. def. 3.

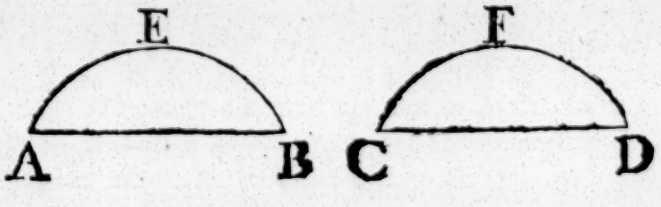
[c] 16. 1.

PROP. XXIV. THEOR.

SIMILAR ſegments of circles upon equal ſtraight lines, are equal to one another. See N.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; the ſegment AEB is equal to the ſegment CFD.

For if the ſegment AEB be applied to the ſegment CFD, ſo as the point A may be on C, and the ſtraight line AB upon CD, the point B ſhall coincide with the point D, becauſe



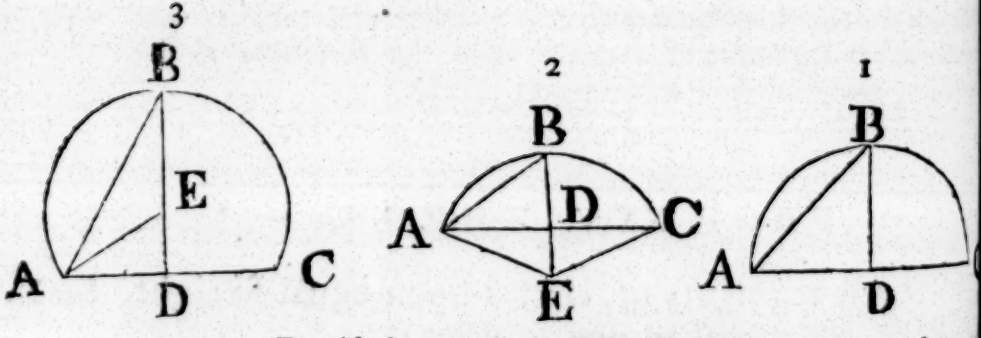
Book III. cauſe AB is equal to CD: Therefore the ſtraight line AB
inciding with CD, the ſegment AEB muſt [a] coincide with
[a] 23. 3. ſegment CFD, and therefore is equal to it. Wherefore ſimi
ſegments, &c. Q. E. D.

PROP. XXV. PROB.

See N. A Segment of a circle being given, to deſcri
the circle of which it is the ſegment.

Let ABC be the given ſegment of a circle; it is required
deſcribe the circle of which it is the ſegment.

[a] 10. 1. Biſect [a] AC in D, and from the point D draw [b] DB at ri
[b] 11. 1. angles to AC, and join AB: Firſt, let the angles ABD, BA
[c] 6. 1. be equal to one another; then the ſtraight line BD is equa
to DA, and therefore to DC: And becauſe the three ſtrai
lines DA, DB, DC are all equal, D is the center of the c
[d] 9. 3. cle [d]: From the center D, at the diſtance of any of the th
DA, DB, DC, deſcribe a circle; this ſhall paſs thro' the oth
points; and the circle of which ABC is a ſegment is deſcribe
And becauſe the center D is in AC, the ſegment ABC is a ſ



micircle: But if the angles ABD, BAD are not equal to o
[c] 23. 1. another, at the point A, in the ſtraight line AB, make [c] the ang
BAE equal to the angle ABD, and produce BD if neceſſary to
and join EC: And becauſe the angle ABE is equal to the ang
BAE, the ſtraight line BE is equal [c] to EA: And becauſe A
is equal to DC, and DE common to the triangles ADE, C
the two ſides AD, DE are equal to the two CD, DE, each
each; and the angle ADE is equal to the angle CDE,
each of them is a right angle; therefore the baſe AE is equ
[f] 4. 1. [f] to the baſe EC: But AE was ſhewn to be equal to EB, wher
fore alſo BE is equal to EC: and the three ſtreight lines

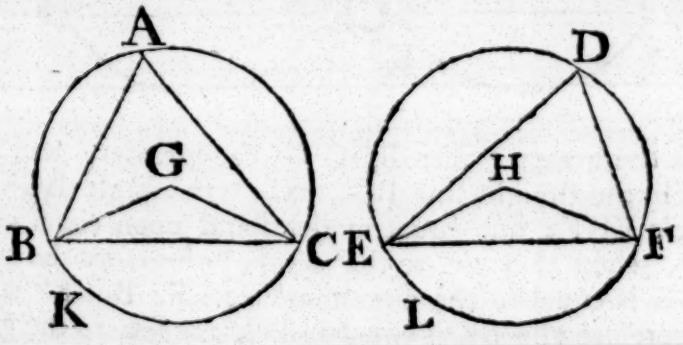
B, EC are therefore equal to one another; wherefore [d] E is Book III.
e center of the circle. From the center E at the diſtance of [d] 9. 3.
y of the three AE, EB, EC deſcribe a circle, this ſhall paſs
ro' the other points; and the circle of which ABC is a ſeg-
ent is deſcribed: And it is evident that if the angle ABD be
reater than the angle BAD, the center E falls without the
gment ABC, which therefore is leſs than a ſemicircle: But
the angle ABD be leſs than BAD, the center E falls within
e ſegment ABC, which is therefore greater than a ſemicircle;
Wherefore a ſegment of a circle being given, the circle is de-
ribed of which it is a ſegment. Which was to be done.

PROP. XXVI. THEOR.

IN equal circles, equal angles ſtand upon equal circumferences, whether they be at the centers, or circumferences.

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centers, and BAC, EDF at their circumferences: the circumference BKC is equal to the circumference ELF.

Join BC, EF; and becauſe the circles ABC, DEF are equal, the ſtraight lines drawn from their centers are equal; therefore the two ſides BG, GC, are equal to the two EH, HF;



and the angle at G is equal to the angle at H; therefore the
baſe BC is equal [a] to the baſe EF: And becauſe the angle at A [a] 4. 1.
is equal to the angle at D, the ſegment BAC is ſimilar [b] to the [b] 11. def.
ſegment EDF: and they are upon equal ſtraight lines BC, EF; 3.
but ſimilar ſegments of circles upon equal ſtraight lines are e-
qual [c] to one another; therefore the ſegment BAC is equal [c] 24. 3.
to the ſegment EDF: But the whole circle ABC is equal to the

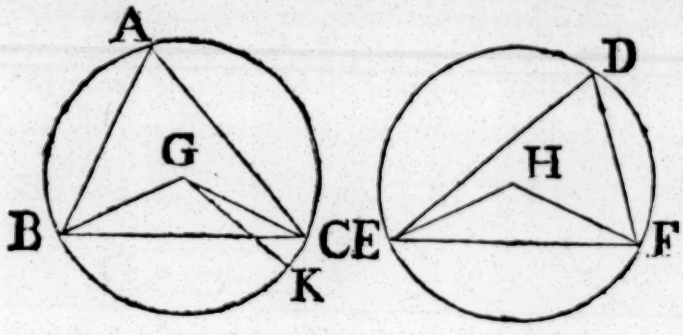
Book III. whole DEF, therefore the remaining ſegment BKC is equal to the remaining ſegment ELF, and the circumference BKC to the circumference ELF. Wherefore in equal circles, &c. Q. E. D.

PROP. XXVII. THEOR.

IN equal circles, the angles which ſtand upon equal circumferences, are equal to one another, whether they be at the centers, or circumferences.

Let the angles BGC, EHF at the centers, and BAC, EDF at the circumferences of the equal circles ABC, DEF ſtand upon the equal circumferences BC, EF: The angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifeſt
[a] 20. 3. [a] that the angle BAC is alſo equal to EDF: But, if not, one



of them is the greater: Let BGC be the greater, and at the
[b] 23. 1. point G, in the ſtraight line BG, make [b] the angle BGK equal to the angle EHF; but equal angles ſtand upon equal circum-
[c] 26. 3. ferences [c], when they are at the center; therefore the circumference BK is equal to the circumference EF: But EF is equal to BC, therefore alſo BK is equal to BC, the leſs to the greater, which is impoſſible: Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it: And the angle at A is half of the angle BGC, and the angle at D half of the angle EHF: Therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

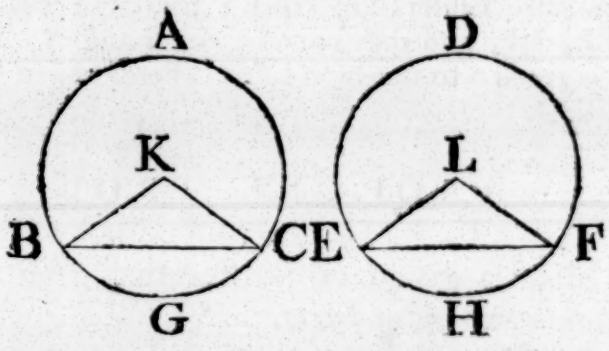
PROP

PROP. XXVIII. THEOR.

IN equal circles, equal ſtraight lines cut off equal circumferences, the greater equal to the greater, and the leſs to the leſs.

Let ABC, DEF be equal circles, and BC, EF equal ſtraight lines in them, which cut off the two greater circumferences BAC, EDF, and the two leſs BGC, EHF: The greater BAC is equal to the greater EDF, and the leſs BGC to the leſs EHF.

Take [a] K, L the centers of the circles, and join BK, KC, EL, LF: And becauſe the circles are equal, the ſtraight lines [a] 1. 3.



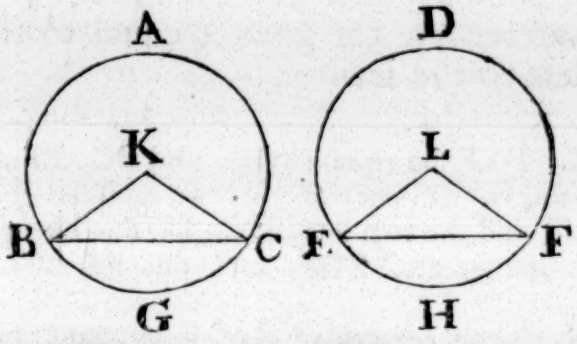
from their centers are equal, therefore BK, KC, are equal to EL, LF; and the baſe BC is equal to the baſe EF; therefore the angle BKC is equal [b] to the angle ELF: But equal angles [b] 8. 1. ſtand upon equal [c] circumferences, when they are at the centers; therefore the circumference BGC is equal to the circumference EHF But the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, viz. BAC is equal to the remaining part EDF. Therefore, in equal circles, &c. Q. E. D. [c] 26. 3.

PROP. XXIX. THEOR.

IN equal circles equal circumferences are ſubtended by equal ſtraight lines.

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF alſo be equal; and join BC, EF: The ſtraight line BC is equal to the ſtraight line EF.

Book III. Take [a] K, L the centers of the circles, and join BK, KC, EL, LF: And becauſe the circumference BGC is equal to the
[a] 1. 3.



[b] 27. 3. circumference EHF, the angle BKC is equal [b] to the angle ELF: And becauſe the circles ABC, DEF are equal, the ſtraight lines from their centers are equal: therefore BK, KC are equal to EL, LF, and they contain equal angles: Therefore the
[c] 4. 1. baſe BC is equal [c] to the baſe EF. Therefore, in equal circles &c. Q. E. D.

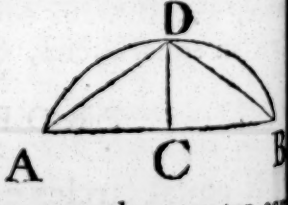
PROP. XXX. PROB.

TO biſect a given circumference, *that is, to divide it into two equal parts.*

Let ADB be the given circumference; it is required to biſect it.

[a] 10. 1. Join AB, and biſect [a] it in C; from the point C draw CD at right angles to AB, and join AD, DB: The circumference ADB is biſected in the point D.

Becauſe AC is equal to CB, and CD common to the two angles ACD, BCD, the two ſides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, becauſe each of them is a right angle; therefore the baſe AD



[b] 4. 1 is equal [b] to the baſe BD: But equal
[c] 28. 3. ſtraight lines cut off equal [c] circumferences, the greater equal to the greater, and the leſs to the leſs, and AD, DB are each of them leſs than a ſemicircle; becauſe DC paſſes through the
[d] Cor. 1. 3. center [d]: Wherefore the circumference AD is equal to the circumference DB: Therefore the given circumference is biſected in D. Which was to be done.

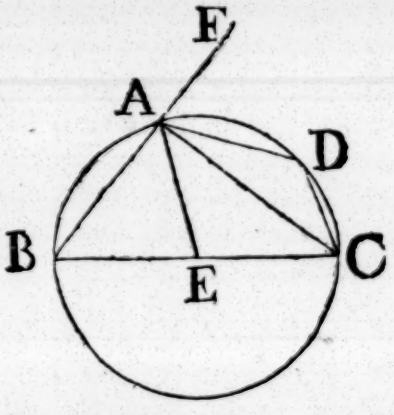
PRO

PROP. XXXI. THEOR.

IN a circle, the angle in a ſemicircle is a right angle; but the angle in a ſegment greater than a ſemicircle is leſs than a right angle; and the angle in a ſegment leſs than a ſemicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and center E; and draw CA dividing the circle into the ſegments ABC, ADC, and join BA, AD, DC; the angle in the ſemicircle BAC is a right angle; and the angle in the ſegment ABC, which is greater than a ſemicircle, is leſs than a right angle; and the angle in the ſegment ADC, which is leſs than a ſemicircle, is greater than a right angle.

Join AE, and produce BA to F; and becauſe BE is equal to EA, the angle EAB is equal [a] to EBA; alſo, becauſe AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB: But FAC, the exterior angle of the triangle ABC, is equal [b] to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right [c] angle: Wherefore the angle BAC in a ſemicircle is a right angle.



[a] 5. 1.

[b] 32. 1.

[c] 10. def. 1.

And becauſe the two angles ABC, BAC of the triangle ABC are together leſs [d] than two right angles, and that BAC is a right angle, ABC muſt be leſs than a right angle; and therefore the angle in a ſegment ABC greater than a ſemicircle, is leſs than a right angle

[d] 17. 1.

And becauſe ABCD is a quadrilateral figure in a circle, any two of its oppoſite angles are equal [e] to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC is leſs than a right angle, wherefore the other ADC is greater than a right angle.

[e] 22. 3.

Beſides, it is manifeſt, that the circumference of the greater ſegment ABC falls without the right angle CAB, but the circumference of the leſs ſegment ADC falls within the right angle CAF. ‘ And this is all that is meant, when in the Greek

Book III

'Greek text, and the tranſlations from it, the angle of the 'greater ſegment is ſaid to be greater, and the angle of the leſs 'ſegment is ſaid to be leſs, than a right angle.'

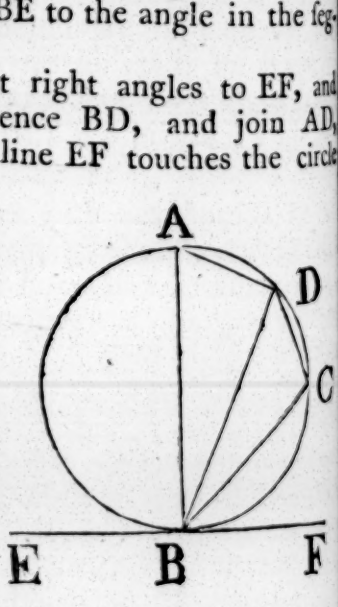
Cor From this it is manifeſt, that if one angle of a triangle be equal to the other two, it is a right angle, becauſe the angle adjacent to it is equal to the ſame two; and when the adjacent angles are equal, they are right angles.

PROP. XXXII. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn cutting the circle, the angles made by this line with the line touching the circle, ſhall be equal to the angles which are in the alternate ſegments of the circle.

Let the ſtraight line EF touch the circle ABCD in B, and from the point B let the ſtraight line BD be drawn cutting the circle: The angles which BD makes with the touching line EF ſhall be equal to the angles in the alternate ſegments of the circle; that is, the angle FBD is equal to the angle which is in the ſegment DAB, and the angle DBE to the angle in the ſegment BCD.

a 11. 1. From the point B draw [a] BA at right angles to EF, and take any point C in the circumference BD, and join AD, DC, CB; and becauſe the ſtraight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact



b 19. 3. B, the center of the circle is [b] in BA; therefore the angle ADB in a ſe-

c 31. 3. micircle is a right [c] angle, and conſequently the other two angles BAD,

d 32. 1. ABD are equal [d] to a right angle: But ABF is likewiſe a right angle; therefore the angle ABF is equal to the angles BAD, ABD: Take from theſe equals the common angle ABD, therefore the remaining angle DBF is equal to the angle BAD which is in the alternate ſegment of the circle; and becauſe ABCD is a quadrilateral

c 22. 3. figure in a circle, the oppoſite angles BAD, BCD are equal [c] to

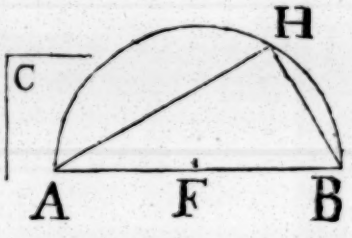
two

wo right angles; therefore the angles DBF, DBE, being like- Book. III.
ife equal [f] to two right angles, are equal to the angles BAD, [f] 13. 1.
CD; and DBF has been proved equal to BAD: therefore the
emaining angle DBE is equal to the angle BCD in the alter-
ate segment of the circle. Wherefore, if a ſtraight line, &c.
Q. E. D.

PROP. XXXIII. PROB.

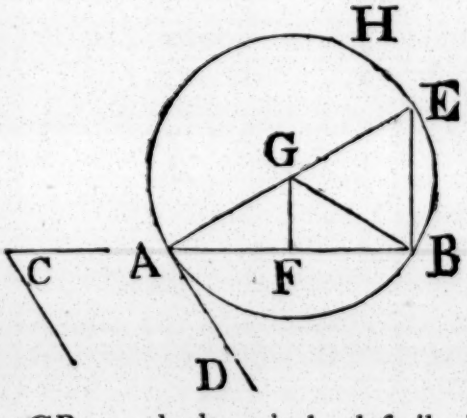
UPON a given ſtraight line to deſcribe a ſegment of a circle, containing an angle equal to a given rectilineal angle. See N.

Let AB be the given ſtraight line, and the angle at C the given rectilineal angle; it is required to deſcribe upon the given ſtraight line AB a ſegment of a circle, containing an angle equal to the angle C.

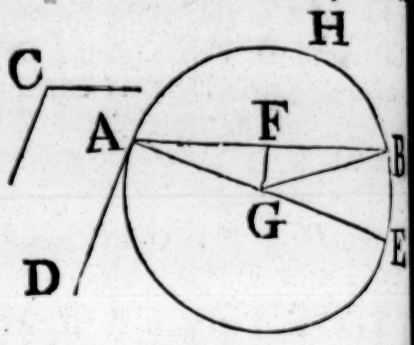


Firſt, Let the angle at C be a right angle, and biſect [a] AB in F, and from the center F, at the diſtance FB, deſcribe the ſemicircle AHB; therefore the angle AHB in a ſemicircle is [b] equal to the right angle at C. [a] 10. 1. [b] 31. 3.

But if the angle C be not a right angle, at the point A in the ſtraight line AB, make [c] the angle BAD equal to the angle C, and from the point A draw [d] AE at right angles to AD; biſect [a] AB in F, and from F draw [d] FG at right angles to AB, and join GB: And becauſe AF is equal to FB, and FG common to the triangles AFG, BFG, the two ſides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the baſe AG is equal [e] to the baſe GB; and the circle deſcribed from the center G, at the diſtance GA, ſhall paſs through the point B; let this be the circle AHB: And becauſe from the point A the extremity of the diameter AE, AD is drawn at right [c] 23. 1. [d] 11. 1. [e] 4. 1.



Book III. right angles to AE, therefore AD [f] touches the circle; and be-
cauſe AB drawn from the point
[f] Cor. 16. 3. of contact A cuts the circle,
the angle DAB is equal to
the angle in the alternate ſeg-
[g] 32. 3. ment AHB [g]: But the angle
DAB is equal to the angle C,
therefore alſo the angle C is
equal to the angle in the ſeg-
ment AHB: Wherefore, upon
the given ſtraight line AB the
ſegment AHB of a circle is deſcribed which contains an angle
equal to the given angle at C. Which was to be done.

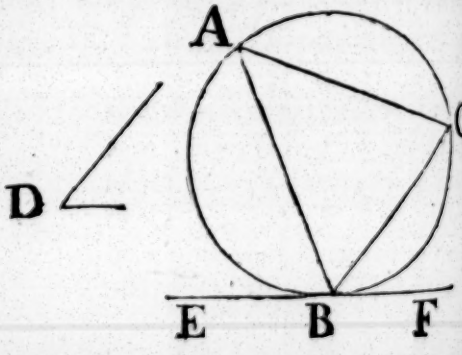


PROP. XXXIV. PROB.

TO cut off a ſegment from a given circle which ſhall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a ſegment from the circle ABC that ſhall contain an angle equal to the angle D.

[a] 17. 3. Draw [a] the ſtraight line EF touching the circle ABC in the point B, and at the point B, in the ſtraight line BF,
[b] 23. 1. make [b] the angle FBC equal to the angle D: Therefore, becauſe the ſtraight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC
[c] 32. 3. is equal [c] to the angle in the alternate ſegment BAC of the circle: But the angle FBC is equal to the angle D; therefore the angle in the ſegment BAC is equal to the angle D: Wherefore the ſegment BAC is cut off from the given circle ABC containing an angle equal to the given angle D: Which was to be done.



PROP.

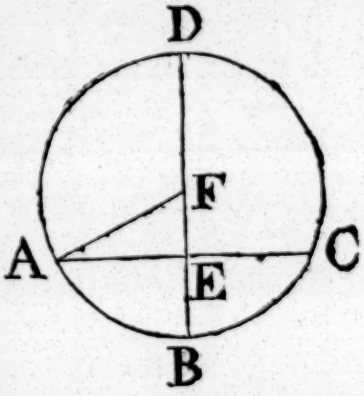
PROP. XXXV. THEOR.

two ſtraight lines within a circle cut one another, the rectangle contained by the ſegments of one them, is equal to the rectangle contained by the ments of the other. See N.

et the two ſtraight lines AC, BD within the circle ABCD, one another in the point E; the rectangle contained by AE, is equal to the rectangle contained by ED.

AC, BD paſs each of them through center, ſo that E is the center; it is ent, that AE, EC, BE, ED being all l, the rectangle AE, EC is likewiſe l to the rectangle BE, ED.

ut let one of them BD paſs through the center, and cut the r AC, which does not paſs through the center, at right angles, e point E: Then, if BD be biſected in F, F is the center he circle ABCD; and join AF: And becauſe BD which s through the center cuts the ſtraight line AC, which does paſs through the centre, at right es in E, AE, EC are equal [a] to another: And becauſe the ſtraight BD is cut into two equal parts in point F, and into two unequal in point E, the rectangle BE, ED ther with the ſquare of EF, is e- [b] to the ſquare of FB; that is, he ſquare of FA; but the ſquares AE, EF are equal [c] to the ſquare A; therefore the rectangle BE, together with the ſquare of EF ual to the ſquares of AE, EF: Take away the common re of EF, and the remaining rectangle BE, ED is equal he remaining ſquare of AE; that is, to the rectangle EC.



[a] 3. 3.

[b] 5. 2.

[c] 47. 1.

ext, Let BD which paſſes through the center, cut the other which does not paſs through the center, in E, but not at t angles: Then, as before, if BD be biſected in F, F is the er of the circle. Join AF, and from F draw [d] FG perpendicular

[d] 12. 1.

Book III.

dicular to AC; therefore AG is equal [a] to GC; where the rectangle AE, EC together with the ſquare of EG is equal [b] to the ſquare of AG: To each of theſe equals add the ſquare of GF, therefore the rectangle AE, EC, together with the ſquares of EG, GF is equal to the ſquares of AG, GF: But the ſquares of EG, GF are equal [c] to the ſquare of EF; and the ſquares of AG, GF are equal to the ſquare of AF: Therefore the rectangle AE, EC together with the ſquare of EF is equal to the ſquare of AF; that is, to the ſquare of FB: But the ſquare of FB is equal [b] to the rectangle BE, ED together with the ſquare of EF; therefore the rectangle AE, EC together with the ſquare of EF, is equal to the rectangle BE, ED together with the ſquare of EF: Take away the common ſquare of EF, and the remaining rectangle AE, EC is therefore equal to the remaining rectangle BE, ED.

[a] 3. 3.
[b] 5. 2.
[c] 47. 1.

Laſtly, Let neither of the ſtraight lines AC, BD paſs through the center: Take the center F, and through E the interſection of the ſtraight lines AC, DB draw the diameter GEFH: And becauſe the rectangle AE, EC is equal, as has been ſhewn, to the rectangle GE, EH; and for the ſame reaſon, the rectangle BE, ED is equal to the ſame rectangle GE, EH; therefore the rectangle AE, EC is equal to the rectangle BE, ED. Wherefore, if two ſtraight lines, Q. E. D.

PROP. XXXVI. THEOR.

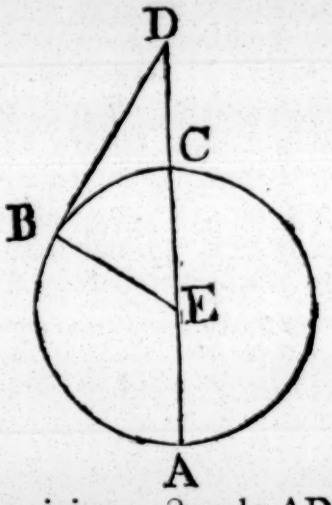
IF from any point without a circle two ſtraight lines be drawn one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, ſhall be equal to the ſquare of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB two ſtraight lines drawn from it, of which DCA cuts the

, and DB touches the ſame: The rectangle AD, DC is to the ſquare of DB

her DCA paſſes through the center, or it does not; firſt, paſs through the center E, and join EB; therefore the angle is a right [a] angle: And becauſe traight line AC is biſected in E, roduced to the point D, the rect- AD, DC together with the e of EC is equal [b] to the ſquare , and CE is equal to EB: There- he rectangle AD, DC together the ſquare of EB is equal to the e of ED: But the ſquare of ED al [c] to the ſquares of EB, BD, be- EBD is a right angle: Therefore ectangle AD, DC together with quare of EB is equal to the es of EB, BD: Take away the non ſquare of EB, therefore the remaining rectangle AD, s equal to the ſquare of the tangent DB.

[a] 18. 3.



[b] 6. 2.

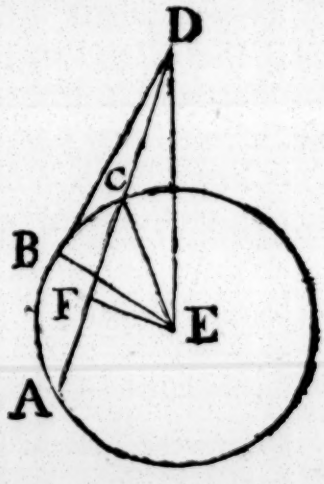
[c] 47. 1.

t if DCA does not paſs through the center of the circle , take [d] the center E, and draw EF perpendicular [e] to and join EB, EC, ED: And becauſe the ſtraight line EF h paſſes through the center, cuts the ſtraight line AC, which not paſs through the center, at right es, it ſhall likewiſe biſect [f] it; there- AF is equal to FC: And becauſe ſtraight line AC is biſected in F, roduced to D, the rectangle AD, together with the ſquare of FC is l [b] to the ſquare of FD: To each of equals add the ſquare of FE, there- the rectangle AD, DC, together the ſquares of CF, FE, is equal to quares of DF, FE: But the ſquare D is equal [c] to the ſquares of DF, becauſe EFD is a right angle; and ſquare of EC is equal to the es of CF, FE; therefore the angle AD, DC, together with the ſquare of EC is e- to the ſquare of ED: And CE is equal to EB; therefore rectangle AD, DC together with the ſquare of EB, is equal to

[d] 1. 3.

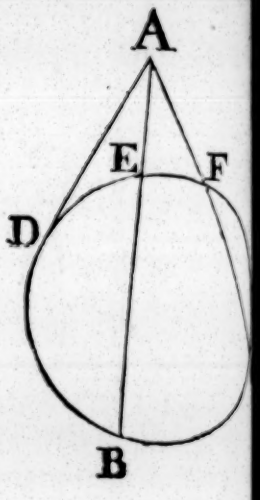
[e] 12. 1.

[f] 3. 3.



Book III. to the ſquare of ED: But the ſquares of EB, BD are equ
the ſquare [c] of ED, becauſe EBD is a right angle; there
[c] 47. 1. the rectangle AD, DC together with the ſquare of EB
qual to the ſquares of EB, BD: Take away the common ſq
of EB, therefore the remaining rectangle AD, DC is equ
the ſquare of DB. Wherefore, if from any point, &c. Q. E.

COR. If from any point without a circle, there be drawn two ſtraight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF: For each of them is equal to the ſquare of the ſtraight line AD which touches the circle.



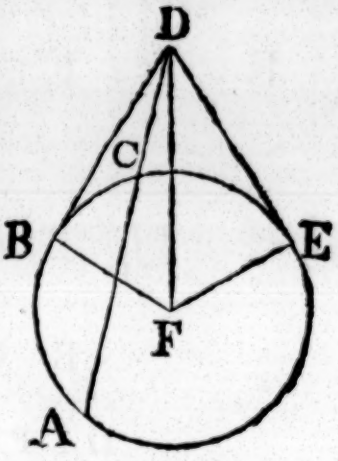
PROP. XXXVII. THEOR.

IF from a point without a circle there be dra
two ſtraight lines, one of which cuts the cir
and the other meets it; if the rectangle contai
by the whole line which cuts the circle, and the p
of it without the circle be equal to the ſquare
the line which meets it, the line which meets
touch the circle.

Let any point D be taken without the circle ABC, and f
it let two ſtraight lines DCA and DB be drawn, of which D
cuts the circle, and DB meets it; if the rectangle AD, DC
equal to the ſquare of DB; DB touches the circle.

[a] 17. 3. Draw [a] the ſtraight line DE touching the circle ABC,
[b] 18. 3. its center F, and join FE, FB, FD; then FED is a right
gle: And becauſe DE touches the circle ABC, and DCA
[c] 36. 3. it, the rectangle AD, DC is equal [c] to the ſquare of DE:
the rectangle AD, DC is, by hypotheſis, equal to the ſq
of DB: therefore the ſquare of DE is equal to the ſquar
DB, and the ſtraight line DE equal to the ſtraight line

FE is equal to FB, wherefore DE, EF are equal to DB, Book III.
and the baſe FD is common to
wo triangles DEF, DBF; there-
he angle DEF is equal [d] to the
DBF, but DEF is a right angle,
fore alſo DBF is a right angle:
FB, if produced, is a diameter,
he ſtraight line which is drawn
ght angles to a diameter, from the
mity of it touches [e] the circle:
refore DB touches the circle ABC.
erefore, if from a point, &c.
E. D.



[d] 8. 1.

[e] 16. 3.

THE

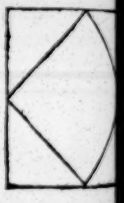
THE ELEMENTS OF EUCLID

BOOK IV.

DEFINITIONS.

I.

A Rectilineal figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure upon the sides of the figure in which it is inscribed, each upon each.

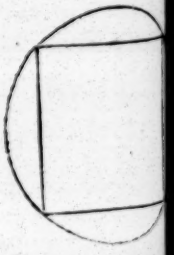


II.

In like manner a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.



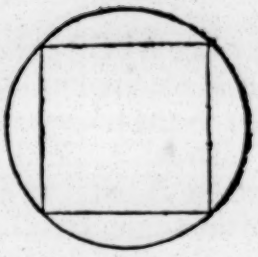
IV.

A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

V.

In like manner a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.



circle is faid to be deſcribed about a rectilineal figure, when the circumference of the circle paſſes through all the angular points of the figure about which it is described.

VII.

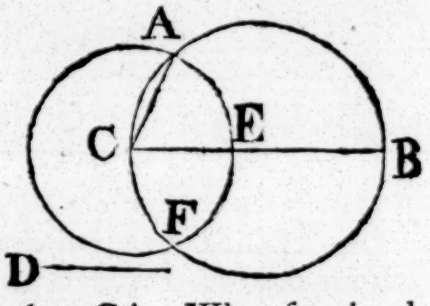
A ſtraight line is faid to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

In a given circle to place a ſtraight line, equal to a given ſtraight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given ſtraight line, not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC the ſtraight line BC is placed equal to D: But if it is not, BC is greater than D; make CE equal [a] to D, and from the centre C, at the diſtance CE, deſcribe the circle AEF, and join CA: Therefore becauſe C is the center of the circle AEF, CA is equal to CE; but D is equal to CE, therefore D is equal to CA: Wherefore in the circle ABC a ſtraight line is placed equal to the given ſtraight line D, which is not greater than the diameter of the circle. Which was to be done.



[a] 3. 1.

PROP. II. PROB.

In a given circle to inſcribe a triangle equiangular to a given triangle.

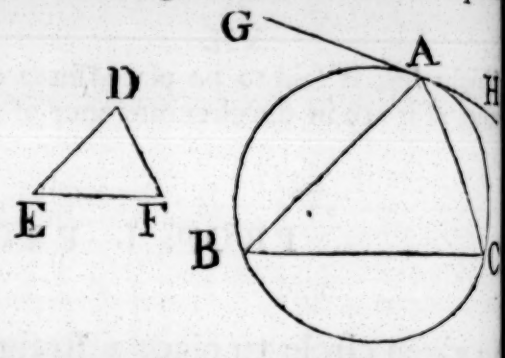
Let

Book IV.

Let ABC be the given circle, and DEF the given triang it is required to inſcribe in the circle ABC a triangle equia lar to the triangle DEF.

a 17. 3. Draw [a] the ſtraight line GAH touching the circle in point A, and at the point A, in the ſtraight line AH, m

b 23. 1. [b] the angle HAC equal to the angle DEF; and at the point in the ſtraight line AG, make the angle GAB equal to the angle DFE, and join BC: Therefore becauſe HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is e-



c 32. 3. qual [c] to the angle ABC in the alternate ſegment of the circle: but HAC is e to the angle DEF, therefore alſo the angle ABC is equa DEF: For the ſame reaſon, the angle ACB is equal to the a

d 32. 1. DFE; therefore the remaining angle BAC is equal [d] to remaining angle EDF: Wherefore the triangle ABC is equ gular to the triangle DEF, and it is inſcribed in the circle A Which was to be done.

PROP. III. PROB.

ABOUT a given circle to deſcribe a triangle eq angular to a given triangle.

Let ABC be the given circle, and DEF the given triang it is required to deſcribe a triangle about the circle ABC e angular to the triangle DEF.

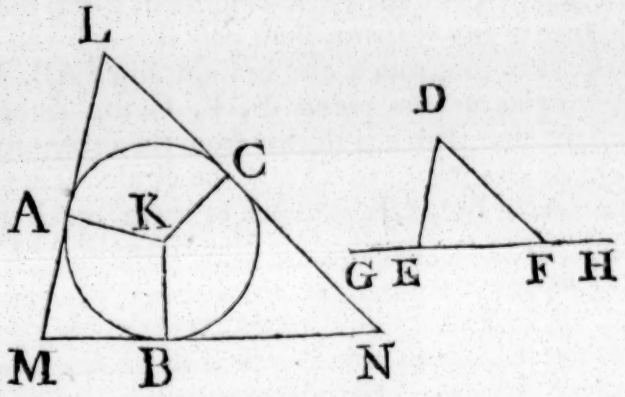
Produce EF both ways to the points G, H, and find center K of the circle ABC, and from it draw any ſtrai

a 23. 1. line KB; at the point K in the ſtraight line KB, make angle BKA equal to the angle DEG, and the angle BK qual to the angle DFH; and through the points A, B, C, d

17. 3. the ſtraight lines LAM, MBN, NCL touching [b] the c ABC: Therefore, becauſe LM, MN, NL touch the circle A in the points A, B, C, to which from the center are dr

c 18. 3. KA, KB, KC, the angles at the points A, B, C, are ri angles: And becauſe the four angles of the quadrilateral

are AMBK are equal to four right angles, for it can be divided into two triangles; and that two of them KAM, KBM are right angles, the other two KB, AMB are ual to two right gles: But the gles DEG, EF are likewise ual [d] to two ght angles; erefore the angles AKB, AMB e equal to the gles DEG



[d] 13. 1.

EF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF: In like manner, the angle LNM may be demonstrated to be equal to FE; and therefore the remaining angle MLN is equal [e] to the remaining angle EDF: Wherefore the triangle LMN is equiangular to the triangle DEF: And it is described about the circle ABC. Which was to be done.

[e] 32. 1.

PROP. IV. PROB.

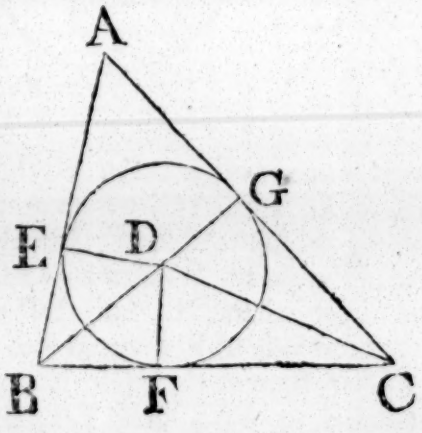
TO inscribe a circle in a given triangle. See N.

Let the given triangle be ABC; it is required to inscribe a circle in ABC.

Bisect [a] the angles ABC, BCA by the straight lines BD, CD meeting one another in the point D, from which draw [b] DE, DF, G perpendiculars to AB, BC, A: And because the angle EBD equal to the angle FBD, for the gle ABC is bisected by BD, and at the right angle BFD is equal the right angle BFD, the two angles EBD, FBD have two gles of the one equal to two gles of the other, and the side D, which is opposite to one of e equal angles in each, is com- on to both; therefore their other es shall be equal [c]; where-

[a] 9. 1.

[b] 12. 1.



[c] 26. 1.

 fore

Book IV. fore DE is equal to DF: For the ſame reaſon, DG is equal t
DF; therefore the three ſtraight lines DE, DF, DG are equal t
one another, and the circle deſcribed from the center D, at th
diſtance of any of them, ſhall paſs through the extremities of th
other two, and touch the ſtraight lines AB, BC, CA, becaus
the angles at the points E, F, G are right angles, and th
ſtraight line which is drawn from the extremity of a diameter a
[d] 16. 3. right angles to it, touches [d] the circle: Therefore the ſtraigh
lines AB, BC, CA do each of them touch the circle, and th
circle EFG is inſcribed in the triangle ABC. Which was t
be done.

PROP. V. PROB.

See N. TO deſcribe a circle about a given triangle.

Let the given triangle be ABC; it is required to deſcribe
circle about ABC.

[a] 10. 1. Biſect [a] AB, AC in the points D, E, and from theſe poin
[b] 11. 1. draw DF, EF at right angles [b] to AB, AC; DF, EF produce

meet one another: For, if they do not meet, they are paralle
wherefore AB, AC, which are at right angles to them, are p
rallel; which is abſurd: Let them meet in F, and join FA; a
ſo, if the point F be not in BC, join BF, CF: Then, becau
AD is equal to DB, and DF common, and at right angles
[c] 4. 1. AB, the baſe AF is equal [c] to the baſe FB: In like manner,
may be ſhewn that CF is equal to FA; and therefore B
is equal to FC; and FA, FB, FC are equal to one a
othe

ner; wherefore the circle deſcribed from the center F, at the tance of one of them, ſhall paſs through the extremities of the er two; and be deſcribed about the triangle ABC, which s to be done.

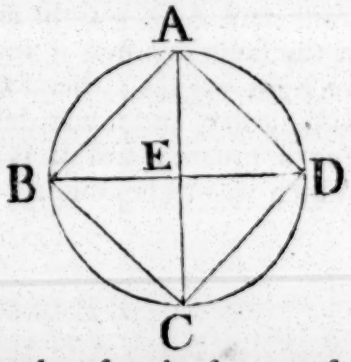
Cor. And it is manifeſt, that when the center of the circle s within the triangle, each of its angles is leſs than a right gle, each of them being in a ſegment greater than a ſemicir-; but, when the center is in one of the ſides of the triangle, angle oppoſite to this ſide, being in a ſemicircle, is a right gle; and, if the center falls without the triangle, the angle poſite to the ſide beyond which it is, being in a ſegment leſs n a ſemicircle, is greater than a right angle: Wherefore, if given triangle be acute angled, the center of the circle falls hin it; if it be a right angled triangle, the center is in the oppoſite to the right angle; and, if it be an obtuſe angled ngle, the center falls without the triangle, beyond the ſide poſite to the obtuſe angle.

PROP. VI. PROB.

TO inſcribe a ſquare in a give circle.

Let ABCD be the given circle; it is required to inſcribe a are in ABCD.

Draw the diameters AC, BD at right angles to one an-er; and join AB, BC, CD, DA; becauſe BE is equal to , for E is the center, and that EA ommon, and at right angles to ; the baſe BA is equal [a] to the AD; and for the ſame reaſon, CD are each of them equal to or AD; therefore the quadri-ral figure ABCD is equilateral alſo rectangular; for the ſtraight BD, being the diameter of the le ABCD, BAD is a ſemicircle;



[a] 4. 1.

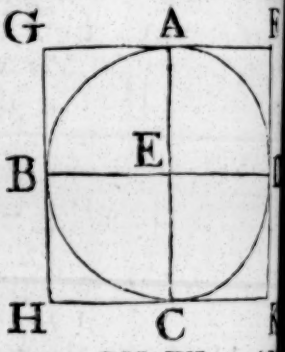
erefore the angle BAD is a right [b] angle; for the ſame reaſon [b] 31. 3.
of the angles ABC, BCD, CDA is a right angle; there-the quadrilateral figure ABCD is rectangular, and it has ſhewn to be equilateral, therefore it is a ſquare; and it is ribed in the circle ABCD. Which was to be done.

PROP. VII. PROB.

TO deſcribe a ſquare about a given circle.

Let ABCD be the given circle; it is required to deſcribe a ſquare about it.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D draw FG, GH, HK, KF touching the circle; and becauſe FG touches the circle ABCD, and EA is drawn from the center E to the point of contact A, the angles at A are right angles; for the ſame reaſon, the angles at the points B, C, D are right angles and becauſe the angle AEB is a right angle, as likewiſe is EBG, GH is parallel to AC; for the ſame reaſon, AC is parallel to FK, and in like manner GF, HK may each of them be demonſtrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and GF is therefore equal to HK, and GH to FK; and becauſe AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK; GH, FK are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is alſo rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB is likewiſe a right angle. In the ſame manner it may be ſhewn that the angles at H, K are right angles; therefore the quadrilateral figure FGHK is rectangular, and it was demonſtrated to be equilateral; therefore it is a ſquare; and it is deſcribed about the circle ABCD. Which was to be done.



[a] 17. 3. [b] 18. 3. [c] 28. 1. [d] 34. 1.

PROP. VIII. PROB.

TO inſcribe a circle in a given ſquare.

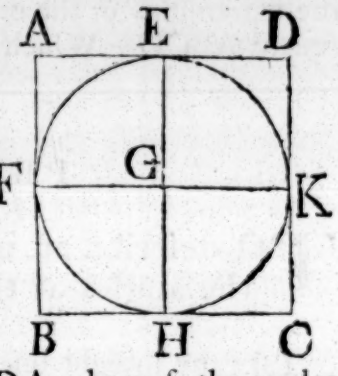
Let ABCD be the given ſquare; it is required to inſcribe a circle in ABCD.

Biſect each of the ſides AB, AD, in the points F, E, through E draw EH parallel to AB or DC, and through F

[a] 10. 1. [b] 31. 1.

K parallel to AD or BC; therefore each of the figures AK, B, AH, HD, AG, GC, BG, GD is a parallelogram, and their pposite ſides are equal [c]; and becauſe AD is equal to AB, and hat AE is the half of AD, and AF the half of AB, AE is equal

[c] 34. 1.



AF; wherefore the ſides oppoſite theſe are equal, viz. FG to GE; in he ſame manner it may be demonrated that GH, GK are each of them qual to FG or GE; therefore the our ſtraight lines GE, GF, GH, GK re equal to one another; and the cirle deſcribed from the center G, at the iſtance of one of them, ſhall paſs thro' he extremities of the other three, and ouch the ſtraight lines AB, BC, CD, DA; becauſe the angles t the points E, F, H, K are right [d] angles, and that the ſtraight ine which is drawn from the extremity of a diameter, at right ngles to it, touches the circle [e]; therefore each of the ſtraight ines AB, BC, CD, DA touches the circle, which therefore is nſcribed in the ſquare ABCD: Which was to be done.

[d] 29. 1.

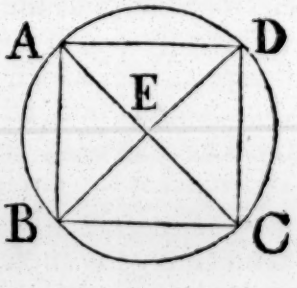
[e] 16. 3.

PROP. IX. PROB.

TO deſcribe a circle about a given ſquare.

Let ABCD be the given ſquare; it is required to deſcribe a ircle about it.

Join AC, BD cutting one another in E; and becauſe DA is qual to AB, and AC common to the triangles DAC, BAC, he two ſides DA, AC are equal to the wo BA, AC; and the baſe DC is equal o the baſe BC; wherefore the angle DAC is equal [a] to the angle BAC, and he angle DAB is biſected by the ſtraight ine AC: In the ſame manner it may be emonſtrated that the angles ABC, CDA re biſected by the ſtraight line BD, and BCA by the ſtraight line AC; therefore, ecauſe the angle DAB is equal to the angle ABC, and that he angle EAB is the half of DAB, and EBA the half of ABC; he angle EAB is equal to the angle EBA; wherefore the ſide EA is equal [b] to the ſide EB: In the ſame manner it may be demon-



[a] 8. 1.

[b] 6. 1.

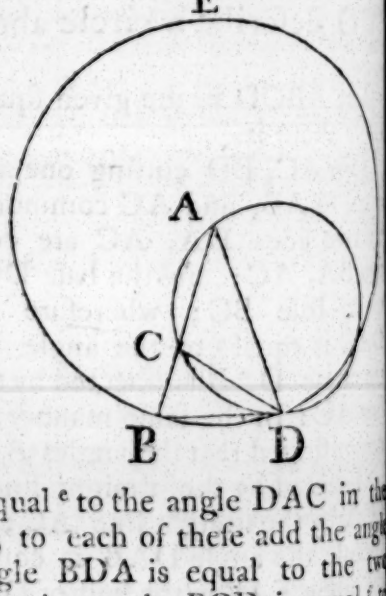
Book IV. demonſtrated that the ſtraight lines EC, ED are each of them equal to EA or EB; therefore the four ſtraight lines EA, EB, EC, ED are equal to one another; and the circle deſcribed from the center E, at the diſtance of one of them, ſhall paſs through the extremities of the other three, and be deſcribed about the ſquare ABCD: Which was to be done.

PROP. X. PROB.

TO deſcribe an iſoſceles triangle, having each of the angles at the baſe double of the third angle.

[a] 11. 2. Take any ſtraight line AB, and divide [a] it in the point C, ſo that the rectangle AB, BC be equal to the ſquare of CA; and from the center A at the diſtance AB, deſcribe the circle BDE; [b] 1. 4. in which place [b] the ſtraight line BD equal to AC, which is not greater than the diameter of the circle BDE; join DA, DC, and [c] 5. 4. about the triangle ADC deſcribe [c] the circle ACD; the triangle ABD is ſuch as is required, that is, each of the angles ABD, ADB is double of the angle BAD.

Becauſe the rectangle AB, BC is equal to the ſquare of AC, and that AC is equal to BD, the rectangle AB, BC is equal to the ſquare of BD, and becauſe from the point B without the circle ACD two ſtraight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC contained by the whole of the cutting line, and the part of it without the circle, is equal to the ſquare of BD which meets it; [d] 37. 3. the ſtraight line BD touches [d] the circle ACD; and becauſe BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal [e] to the angle DAC in the [e] 32. 3. alternate ſegment of the circle; to each of theſe add the angle CDA; therefore the whole angle BDA is equal to the two angles CDA, DAC; but the exterior angle BCD is equal [f] to [f] 32. 1. the angles CDA, DAC; therefore alſo BDA is equal to BCD; but



but BDA is equal [g] to the angle CBD, becauſe the ſide AD is equal to the ſide AB; therefore CBD, or DBA is equal to BCD; and conſequently the three angles BDA, DBA, BCD, are equal to one another; and becauſe the angle DBC is equal to the angle BCD, the ſide BD is equal [g] to the ſide DC; but BD was made equal to CA; therefore alſo CA is equal to CD, and the angle CDA equal [g] to the angle DAC; therefore the angles CDA, DAC together, are double of the angle DAC: But BCD is equal to the angles CDA, DAC; therefore alſo BCD is double of DAC, and BCD is equal to each of the angles BDA, DBA; each therefore of the angles BDA, DBA is double of the angle DAB; wherefore an iſoſceles triangle ABD is deſcribed, having each of the angles at the baſe double of the third angle: Which was to be done. Book IV.

[g] 5. 1.

[h] 6. 1.

PROP. XI. PROB.

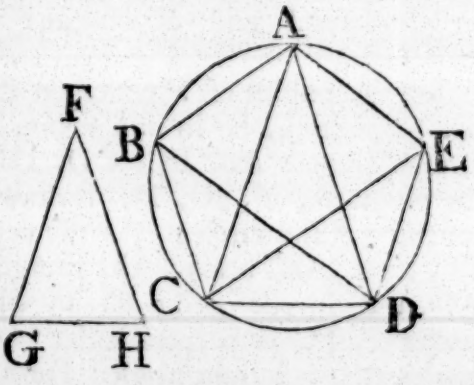
TO inſcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle; it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

Deſcribe [a] an iſoſceles triangle FGH, having each of the angles at G, H double of the angle at F; and in the circle ABCDE inſcribe [b] the triangle ACD equiangular to the triangle FGH, ſo that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. Biſect [c] the angles ACD, CDA by the ſtraight lines CE, DB; and join AB, BC, DE, EA. ABCDE is the pentagon required.

[a] 10. 4.

[b] 2. 4.



[c] 9. 1.

Becauſe each of the angles ACD, CDA is double of CAD, and are biſected by the ſtraight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but equal angles ſtand upon equal [d] circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another,

[d] 26. 3.

Book IV.

another: And equal circumferences are ſubtended by equal ſtraight lines; therefore the five ſtraight lines AB, BC, CD, DE, EA are equal to one another. Wherefore the pentagon ABCDE is equilateral. It is alſo equiangular; becauſe the circumference AB is equal to the circumference DE: If to each be added BCD, the whole ABCD is equal to the whole EDCB. And the angle AED ſtands on the circumference ABCD, and the angle BAE on the circumference EDCB; therefore the angle BAE is equal to the angle AED: For the ſame reaſon, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED: Therefore the pentagon ABCDE is equiangular; and it has been ſhewn that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inſcribed. Which was to be done.

[c] 29. 3.

[f] 27. 3.

PROP. XII. PROB.

TO deſcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be given circle; it is required to deſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon inſcribed in the circle, by the laſt propoſition, be in the points A, B, C, D, E, ſo that the circumferences AB, BC, CD, DE, EA are equal; and through the points A, B, C, D, E draw GH, HK, KL, LM, MG touching the circle; take the center F, and join FB, FK, FC, FL, FD: And becauſe the ſtraight line KL touches the circle ABCDE in the point C, to which FC is drawn from the center F, FC is perpendicular to KL; therefore each of the angles at C is a right angle: For the ſame reaſon, the angles at the points B, D are right angles: And becauſe FCK is a right angle, the ſquare of FK is equal to the ſquares of FC, CK: For the ſame reaſon, the ſquare of FK is equal to the ſquares of FB, BK: Therefore the ſquares of FC, CK are equal to the ſquares of FB, BK, of which the ſquare of FC is equal to the ſquare of FB; the remaining ſquare of CK is therefore equal to the

[a] 11. 4.

[b] 17. 3.

[c] 18. 3

[d] 47. 1.

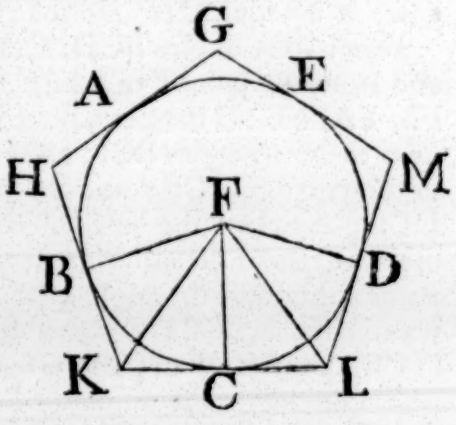
remaining ſquare of BK, and the ſtraight line CK equal to : And becauſe FB is equal to FC, and FK common to the ngles BFK, CFK, the two BF, FK are equal to the two CF, ; and the baſe BK is equal to the baſe KC; therefore the le BFK is equal [c] to the angle KFC, and the angle BKF to C: Wherefore the angle BFC is double of the angle KFC, BKC double of FKC: For the ſame reaſon, the angle CFD ouble of the angle CFL, and CLD double of CLF: And beſe the circumference BC is equal to the circumference CD, the

Book IV.

[c] 8. 1.

le BFC is equal [f] to the an- CFD; And BFC is double the angle KFC, and CFD ble of CFL; therefore the le KFC is equal to the an- CFL; and the right angle K is equal to the right angle L: Therefore, in the two ngles FKC, FLC, there are angles of one equal to two les of the other, each to h, and the ſide FC, which

[f] 27. 3.



djacent to the equal angles in each, is common to both; there- the other ſides ſhall be equal [g] to the other ſides, and the third le to the third angle: Therefore the ſtraight line KC is equal to , and the angle FKC to the angle FLC: And becauſe KC is al to CL, KL is double of KC: In the ſame manner, it may hewn that HK is double of BK: And becauſe BK is equal to , as was demonſtrated, and that KL is double of KC, and HK ble of BK, HK ſhall be equal to KL: In like manner it may hewn that GH, GM, ML are each of them equal to HK or : Therefore the pentagon GHKLM is equilateral. It is alſo iangular; for, ſince the angle FKC is equal to the angle FLC, that the angle HKL is double of the angle FKC, and KLM ble of FLC, as was before demonſtrated; the angle HKL is al to KLM: And in like manner it may be ſhewn, that each he angles KHG, HGM, GML is equal to the angle HKL or M: Therefore the five angles GHK, HKL, KLM, LMG, H being equal to one another, the pentagon GHKLM is e- ngular: And it is equilateral, as was demonſtrated; and it ſcribed about the circle ABCDE. Which was to be done.

[g] 26. 1.

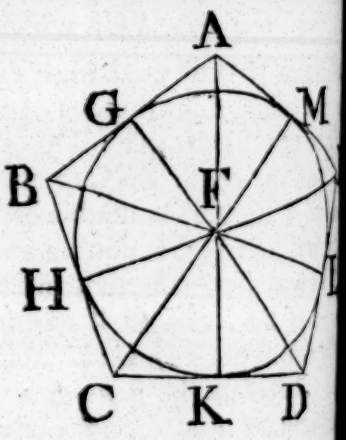
PROP.

PROP. XIII. PROB.

TO inſcribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inſcribe a circle in the pentagon ABCDE.

[a] 9. 1. Biſect [a] the angles BCD, CDE by the ſtraight lines CF, DF, and from the point F in which they meet draw the ſtraight lines FB, FA, FE: Therefore, ſince BC is equal to CD, and CF common to the triangles BCF, DCF, the two ſides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle [b] 4. 1. DCF; therefore the baſe BF is equal [b] to the baſe FD, and the other angles to the other angles, to which the equal ſides are oppoſite; therefore the angle CBF is equal to the angle CDF: And becauſe the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is alſo double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is biſected by the ſtraight line BF: In the ſame manner it may be demonſtrated, that the angles BAE, AED are biſected by the ſtraight lines AF, [c] 12. 1. FE: From the point F draw [c] FG, FH, FK, FL, FM perpendiculars to the ſtraight lines AB, BC, CD, DE, EA: And becauſe the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC, in the triangles FHC, FKC there are two angles of one equal to two angles of the other, and the ſide FC, which is oppoſite to one of the equal angles in each, is common to both; therefore [d] 26. 1. the other ſides ſhall be equal [d], each to each; wherefore the perpendicular FH is equal to the perpendicular FK: In the ſame manner it may be demonſtrated that FL, FM, FG are each of them equal to FH or FK; therefore the five ſtraight lines, FG, FH, FK, FL, FM are equal to one another: Wherefore the circle deſcribed from the center F, at the diſtance of one of the five, ſhall paſs through the extremities of the other four, and

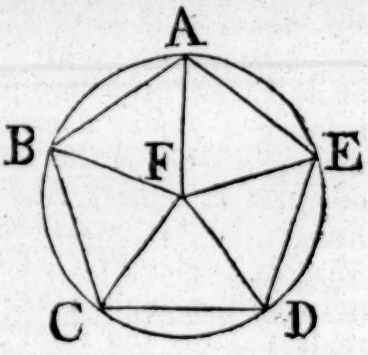


ch the ſtraight lines AB, BC, CD, DE, EA, becauſe the Book IV.
les at the points G, H, K, L, M are right angles; and that
raight line drawn from the extremity of the diameter of a
cle at right angles to it, touches [c] the circle: Therefore each [c] 16. 3.
he ſtraight lines AB, BC, CD, DE, EA touches the circle;
erefore it is inſcribed in the pentagon ABCDE. Which
s to be done.

PROP. XIV. PROB.

TO deſcribe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pen-
on; it is required to deſcribe a circle about it.
Biſect [a] the angles BCD, CDE by the ſtraight lines CF, FD, [a] 9. 1.
d from the point F in which they meet draw the ſtraight
es FB, FA. FE to the points B,
E. It may be demonſtrated, in
ſame manner as in the preceeding
poſition, that the angles CBA,
AE, AED are biſected by the
ight lines FB, FA. FE: And
auſe the angle BCD is equal to
angle CDE, and that FCD is
half of the angle BCD, and CDF
half of CDE; the angle FCD is
al to FDC; wherefore the ſide
is equal [b] to the ſide FD: In like manner it may be demon- [b] 6. 1.
ted that FB, FA, FE are each of them equal to FC or FD:
erefore the five ſtraight lines FA, FB, FC, FD, FE are e-
l to one another; and the circle deſcribed from the center
at the diſtance of one of them, ſhall paſs through the extre-
ties of the other four, and be deſcribed about the equilateral
equiangular pentagon ABCDE. Which was to be done.



PROP.

PROP. XV. PROB.

See N. **TO inſcribe an equilateral and equiangular hexagon in a given circle.**

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in it.

Find the center G of the circle ABCDEF, and draw the diameter AGD; and from D as a center, at the diſtance DG deſcribe the circle EGCH, join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA: The hexagon ABCDEF is equilateral and equiangular.

Becauſe G is the center of the circle ABCDEF, GE is equal to GD: And becauſe D is the center of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one another, becauſe the angles at [a] 5. 1. the baſe of an iſoſceles triangle are equal [a]: And the three angles [b] 32. 1. of a triangle are equal [b] to two right angles; therefore the angle EGD is the third part of two right angles: In the ſame manner it may be demonſtrated that the angle DGC is alſo the third part of two right angles: And becauſe the ſtraight line GC makes with EB the [c] 13. 1. adjacent angles EGC, CGB equal [c] to two right angles; the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB are equal to one [d] 15 1. another: And to theſe are equal [d] the vertical oppoſite angles BGA, AGF, FGE: Therefore the ſix angles EGD DGC, CGB, BGA, AGF, FGE, are equal to one another: But equal [e] 26. 3. angles ſtand upon equal [e] circumferences; therefore the ſix circumferences AB, BC, CD, DE, EF, FA are equal to one another: [f] 9. 3. And equal circumferences are ſubtended by equal [f] ſtraight lines; therefore the ſix ſtraight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is alſo equiangular; for, ſince the circumference AF is equal to ED, to each of theſe add the circumference ABCD; therefore the whole circumference FABCD ſhall be equal to the whole EDCB

nd the angle FED ſtands upon the circumference FABCD, Book IV.
d the angle AFE upon EDCBA; therefore the angle AFE is
ual to FED: In the ſame manner it may be demonſtrated that
other angles of the hexagon ABCDEF are each of them
ual to the angle AFE or FED: Therefore the hexagon is e-
iangular: And it is equilateral, as was ſhewn; and it is in-
ibed in the given circle ABCDEF. Which was to be done.

Cor. From this it is manifeſt, that the ſide of the hexagon
equal to the ſtraight line from the center, that is, to the ſe-
diameter of the circle.

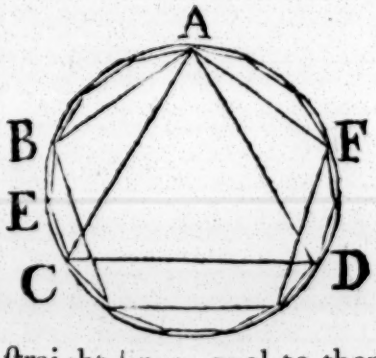
And if thro' the points A, B, C, D, E, F there be drawn
aight lines touching the circle, an equilateral and equiangu-
hexagon ſhall be deſcribed about it, which may be demon-
ated from what has been ſaid of the pentagon; and likewiſe a
cle may be inſcribed in a given equilateral and equiangular
xagon, and circumſcribed about it, by a method like to that
d for the pentagon.

PROP. XVI. PROB.

TO inſcribe an equilateral and equiangular quinde- See N.
cagon in a given circle.

Let ABCD be the given circle; it is required to inſcribe an
uilateral and equiangular quindecagon in the circle ABCD.

Let AC be the ſide of an equilateral triangle inſcribed [a] in [a] 2. 4.
e circle, and AB the ſide of an equilateral and equiangular
ntagon inſcribed [b] in the ſame; therefore, of ſuch equal parts [b] 11. 4.
the whole circumference ABCDF contains fifteen, the cir-
mference ABC, being the third
rt of the whole, contains five;
d the circumference AB, which
the fifth part of the whole, con-
ins three; therefore BC their dif-
rence contains two of the ſame
arts: Biſect [c] BC in E; therefore [c] 30. 3.
E, EC are, each of them, the fif
enth part of the whole circumfe
ence ABCD: Therefore, if the
raight lines BE, EC be drawn and ſtraight lines equal to them
e placed [d] around in the whole circle, an equilateral and equian- [d] 1. 4.
ular quindecagon ſhall be inſcribed in it. Which was to be
one.



And

Book IV.

And in the ſame manner as was done in the pentagon, i
through the points of diviſion made by inſcribing the quinde
gon, ſtraight lines be drawn touching the circle, an equilate
and equiangular quindecagon ſhall be deſcribed about it: A
likewiſe, as in the pentagon, a circle may be inſcribed in
given equilateral and equiangular quindecagon, and circu
ſcribed about it.

THE ELEMENTS OF EUCLID.

BOOK V.

DEFINITIONS.

I.

LESS magnitude is said to be a part of a greater magnitude, when the less measures the greater, that is, 'when the less is contained a certain number of times exactly in the greater.'

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, 'when the greater contains the less a certain number of times exactly.'

III.

A ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity." See N.

IV.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or,

Book V.

or, if the multiple of the first be greater than that of the
cond, the multiple of the third is also greater than that of
fourth.

VI.

Magnitudes which have the same ratio are called proportion
N. B. 'When four magnitudes are proportionals, it is u
'ally expressed by saying, the first is to the second, as
'third to the fourth.'

VII.

When of the equimultiples of four magnitudes (taken as in
5th definition) the multiple of the first is greater than
of the second, but the multiple of the third is not gre
than the multiple of the fourth; then the first is said to h
to the second a greater ratio than the third magnitude ha
the fourth; and, on the contrary, the third is said to hav
the fourth a less ratio than the first has to the second.

VIII.

"Analogy, or proportion, is the similitude of ratios."

IX.

Proportion consists in three terms at least.

X.

When three magnitudes are proportionals, the first is sai
have to the third the duplicate ratio of that which it ha
the second.

XI.

See N. When four magnitudes are continual proportionals, the fir
said to have to the fourth the triplicate ratio of that whic
has to the second, and so on, quadruplicate, &c. increas
the denomination still by unity, in any number of prop
tionals.

Definition A, to wit, of compound ratio.

When there are any number of magnitudes of the same ki
the first is said to have to the last of them the ratio co
pounded of the ratio which the first has to the second, and
the ratio which the second has to the third, and of the r
which the third has to the fourth, and so on unto the
magnitude.

For example, If A, B, C, D be four magnitudes of the sa
kind, the first A is said to have to the last D the ratio co
pounded of the ratio A to B, and of the ratio of B to C,
of the ratio of C to D; or, the ratio of A to D is said
be compounded of the ratios of A to B, B to C, and C to

nd if A has to B, the ſame ratio which E has to F; and B to C, the ſame ratio that G has to H; and C to D, the ſame that K has to L; then, by this definition, A is ſaid to have to D the ratio compounded of ratios which are the ſame with the ratios of E to F, G to H, and K to L: And the ſame thing is to be underſtood when it is more briefly expreſſed, by ſaying A has to D, the ratio compounded of the ratios of E to F, G to H, and K to L. Book V.

like manner, the ſame things being ſuppoſed, if M has to N the ſame ratio which A has to D; then, for ſhortneſs ſake, M is ſaid to have to N, the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

proportionals, the antecedent terms are called homologous to one another, as alſo the conſequents to one another.

Geometers make uſe of the following technical words to ſig-'nify certain ways of changing either the order or magni-'tude of proportionals, ſo as that they continue ſtill to be 'proportionals.'

XIII.

mutando, or alternando, by permutation, or alternately; this word is uſed when there are four proportionals, and it is inferred, that the firſt has the ſame ratio to the third, which the ſecond has to the fourth; or that the firſt is to the third, as the ſecond to the fourth: As is ſhewn in the 16th prop. of this 5th book. See N.

XIV.

vertendo, by inverſion: when there are four proportionals, and it is inferred, that the ſecond is to the firſt, as the fourth to the third. Prop. B. book 5.

XV.

mponendo, by compoſition; when there are four proporti-onals, and it is inferred, that the firſt, together with the ſe-cond, is to the ſecond, as the third, together with the fourth, is to the fourth. 18th prop. book 5.

XVI.

ividendo, by diviſion; when there are four proportionals, and it is inferred, that the exceſs of the firſt above the ſecond, is to the ſecond, as the exceſs of the third above the fourth, is to the fourth. 17th prop. book 5

XVII.

nvertendo, by converſion; when there are four proportion-

Book V.

als, and it is inferred, that the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth. Prop. E. book 5.

XVIII.

Ex aequali (ſc. diſtantia), or, ex aequo, from equality of diſtance; when there is any number of magnitudes more than two, and as many others, ſo that they are proportionals when taken two and two of each rank, and it is inferred, that the firſt is to the laſt of the firſt rank of magnitudes as the firſt is to the laſt of the others: 'Of this there are the two 'following kinds, which ariſe from the different order in 'which the magnitudes are taken two and two.'

XIX.

Ex aequali, from equality; this term is uſed ſimply by itſelf, when the firſt magnitude is to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and as the ſecond is to the third of the firſt rank, ſo is the ſecond to the third of the other; and ſo on in order, and the inference is as mentioned in the preceeding definition; whence this is called ordinate proportion. It is demonſtrated in 22d prop. book 5.

XX.

Ex aequali, in proportione perturbata, ſeu inordinata; from equality, in perturbate or diſorderly proportion †; this term is uſed when the firſt magnitude is to the ſecond of the firſt rank, as the laſt but one is to the laſt of the ſecond rank; and as the ſecond is to the third of the firſt rank, ſo is the laſt but two to the laſt but one of the ſecond rank; and as the third is to the fourth of the firſt rank, ſo is the third from the laſt to the laſt but two of the ſecond rank; and ſo on in a croſs order: And the inference is as in the 18th definition. It is demonſtrated in 23d prop. of book 5.

AXIOMS.

I.

EQUIMULTIPLES of the ſame, or of equal magnitudes, are equal to one another.

II. Thoſe

† 4. Prop. lib. 2. Archimedis de ſphaera et cylindro.

II.

Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

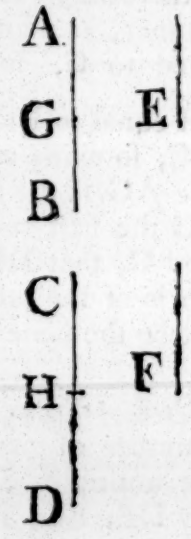
That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each; whatsoever multiple AB is of E, the same multiple shall AB and CD together be of E and F together.

Because AB is the same multiple of E that CD is of F, as many magnitudes as are in AB equal to E, so many are there in CD equal to F. Divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD equal each of them to F: The number therefore of the magnitudes CH, HD shall be equal to the number of the others AG, GB: And because AG is equal to E, and CH to F, therefore AG and CH together are equal to [a] E and F together: For the same reason, because GB is equal to E, and HD to F; GB and HD together are equal to E and F together. Wherefore, as many magnitudes as are in AB equal to E, so many are there in AB, CD together equal to E and F together. Therefore, whatsoever multiple AB is of E, the same multiple is AB and CD together of E and F together.



[a] Ax. 2. 1.

Therefore, if any magnitudes, how many soever, be equimultiples of as many, each of each, whatsoever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the other: For the same demonstration

Book. V. 'holds in any number of magnitudes, which was here applied 'to two.' Q. E. D.

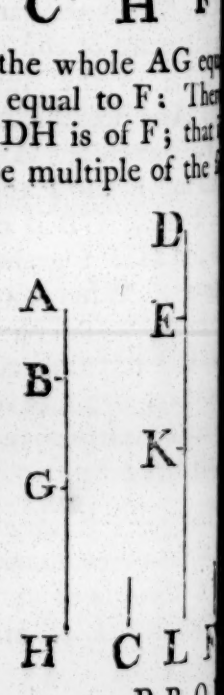
PROP. II. THEOR.

IF the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let AB the first be the same multiple of C the second, that DE the third is of F the fourth; and BG the fifth the same multiple of C the second, that EH the sixth is of F the fourth.

Then is AG the first together with the fifth the same multiple of C the second, that DH the third together with the sixth is of F the fourth.

Because AB is the same multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F: In like manner, as many as there are in BG equal to C, so many are there in EH equal to F: As many then as are in the whole AG equal to C, so many are there in the whole DH equal to F: Therefore AG is the same multiple of C, that DH is of F; that is, AG the first and fifth together, is the same multiple of the second C, that DH the third and sixth together is of the fourth F. If, therefore, the first be the same multiple, &c. Q. E D.



Cor. 'From this it is plain, that if any 'number of magnitudes AB, BG, GH, 'be multiples of another C; and as ma- 'ny DE, EK, KL be the same multiples 'of F, each of each; the whole of the 'first, viz. AH, is the same multiple of 'C, that the whole of the last, viz. DL, 'is of F.'

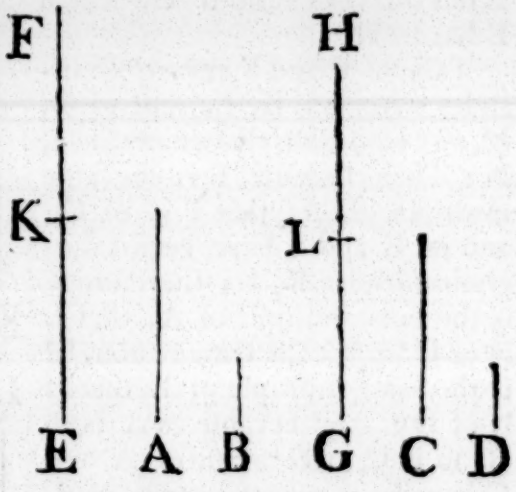
PRO

PROP. III. THEOR.

IF the firſt be the ſame multiple of the ſecond, which the third is of the fourth; and if of the firſt and third there be taken equimultiples, theſe ſhall be equimultiples the one of the ſecond, and the other of the fourth.

Let A the firſt be the ſame multiple of B the ſecond, that C the third is of D the fourth; and of A, C let the equimultiples EF, GH be taken: Then EF is the ſame multiple of B, that GH is of D.

Becauſe EF is the ſame multiple of A, that GH is of C, there are as many magnitudes in EF equal to A, as are in GH equal to C: Let EF be divided into the magnitudes EK, KF, each equal to A, and GH into GL, LH, each equal to C: The number therefore of the magnitudes EK, KF, ſhall be equal to the number of the others GL, LH: And becauſe A is the ſame multiple of B, that C is of D, and that EK is equal to A, and GL to C; therefore EK is the ſame multiple of B, that GL is of D: For the ſame reaſon, KF is the ſame multiple of B, that LH is of D; and ſo, if there be more parts in EF, GH equal to A, C: Becauſe, therefore, the firſt EK is the ſame multiple of the ſecond B, which the third GL is of the fourth D, and that the fifth KF is the ſame multiple of the ſecond B, which the ſixth LH is of the fourth D; EF the firſt together with the fifth is the ſame multiple [a] of the ſecond B, which GH the third together with the ſixth is of the fourth D. If, therefore, the firſt, &c. Q. E. D.

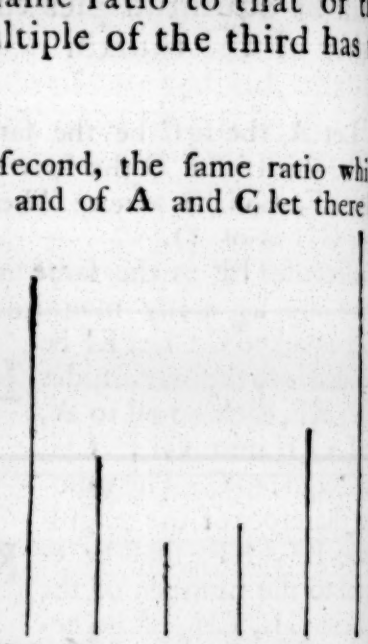


[a] 2 5.

PROP. IV. THEOR.

See N. IF the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. 'the equimultiple 'of the first shall have the same ratio to that of the 'second, which the equimultiple of the third has to 'that of the fourth.'

Let A the first have to B the second, the same ratio which the third C has to the fourth D; and of A and C let there be taken any equimultiples whatever E, F; and of B and D any equimultiples whatever G, H: Then E has the same ratio to G, which F has to H.



Take of E and F any equimultiples whatever K, L, and of G, H, any equimultiples whatever M, N: Then, because E is the same multiple of A, that F is of C; and of E and F have been taken equimultiples K, L; therefore K is the same multiple of A, that L is of C[a]: For the same reason, M is the same multiple of B, that N is of D: And because as A is to B, so is C to D[b], and of A and C have been taken certain equimultiples K, L; and of B and D have been taken certain equimultiples M, N; if therefore K be greater than M, L is greater than N; and if equal, equal; if less, less[c]. And K, L are any equimultiples whatever of E, F; and M, N any whatever of G, H: As therefore E is to G, so is[c] F to H. Therefore, if the first, &c. Q. E. D.

[a] 3. 5.

[b] Hypoth.

[c] 5. def. 5.

Cor. Likewise, if the first has the same ratio to the second, which the third has to the fourth, then also any equimulti-

ples

ples whatever of the first and third have the same ratio to the second and fourth: And, in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth. Book V.

Let A the first have to B the second, the same ratio which the third C has to the fourth D, and of A and C let E and F be any equimultiples whatever; then E is to B, as F to D.

Take of E, F any equimultiples whatever K, L, and of B, D any equimultiples whatever G, H; then it may be demonstrated, as before, that K is the same multiple of A, that L is of C; And because A is to B, as C is to D, and of A and C certain equimultiples have been taken, viz. K and L; and of B and D certain equimultiples G, H; therefore, if K be greater than G, L is greater than H; and if equal, equal; if less, less [c]: And K, L are any equimultiples of E, F, and G, H any whatever of B, D; as therefore E is to B, so is F to D: And in the same way the other case is demonstrated. [c] 5. def. 5.

PROP. V. THEOR.

IF one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole. See N.

Let the magnitude AB be the same multiple of CD, that AE taken from the first, is of CF taken from the other; the remainder EB shall be the same multiple of the remainder FD, that the whole AB is of the whole CD.

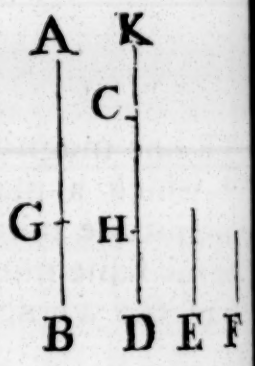
Take AG the same multiple of FD, that AE is of CF: Therefore AE is [a] the same multiple of CF, that EG is of CD: But AE, by the hypothesis, is the same multiple of CF, that AB is of CD: Therefore EG is the same multiple of CD that AB is of CD; wherefore EG is equal to AB [b]: Take from them the common magnitude AE; the remainder AG is equal to the remainder EB. Wherefore, since AE is the same multiple of CF, that AG is of FD, and that AG is equal to EB; therefore AE is the same multiple of CF, that EB is of FD: But AE is the same multiple of CF, [a] 1. 5. [b] 1. Ax. 5.

Book V. that AB is of CD; therefore EB is the ſame multiple of FD, that AB is of CD. Therefore, if one magnitude, &c. Q. E. D.

PROP. VI. THEOR.

See N. IF two magnitudes be equimultiples of two others, and if equimultiples of theſe be taken from the firſt two, the remainders are either equal to theſe others, or equimultiples of them.

Let the two magnitudes AB, CD be equimultiples of the two E, F, and AG, CH taken from the firſt two be equimultiples of the ſame E, F; the remainders GB, HD are either equal to E, F, or equimultiples of them.



Firſt, Let GB be equal to E; HD is equal to F: Make CK equal to F; and becauſe AG is the ſame multiple of E, that CH is of F, and that GB is equal to E, and CK to F; therefore AB is the ſame multiple of E, that KH is of F. But AB, by the hypotheſis, is the ſame multiple of E that CD is of F; therefore KH is the ſame multiple of F, that CD is of F; [a] 1. Ax. wherefore KH is equal to CD [a]: Take away the common magnitude CH, then the remainder KC is equal to the remainder HD: But KC is equal to F; HD therefore is equal to F.

But let GB be a multiple of E; then HD is the ſame multiple of F: Make CK the ſame multiple of F, that GB is of E: And becauſe AG is the ſame multiple of E, that CH is of F, and GB the ſame multiple of E, that CK is of F; therefore AB is the ſame multiple of E, that KH is of [b] 2. 5. F [b]: But AB is the ſame multiple of E, that CD is of F; therefore KH is the ſame multiple of F, that CD is of it; wherefore KH is equal to CD [a]: Take away CH from both; therefore the remainder KC is equal to the remainder HD: And becauſe GB is the ſame multiple of E, that KC is of F, and that KC is equal to HD; therefore HD is the ſame multiple of F, that GB is of E: If, therefore, two magnitudes, &c. Q. E. D.

PROP.

PROP. A. THEOR.

F the firſt of four magnitudes has to the ſecond, the ſame ratio which the third has to the fourth; en, if the firſt be greater than the ſecond, the third lſo greater than the fourth; and, if equal, equal; leſs, leſs. See N.

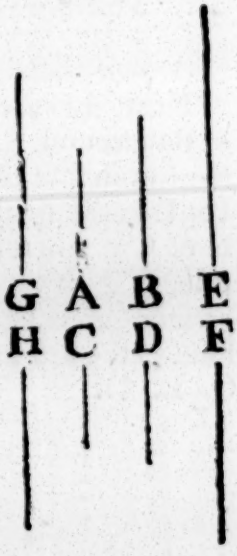
ake any equimultiples of each of them, as the doubles of ; then, by def. 5th of this book, if the double of the firſt be ter than the double of the ſecond, the double of the third is ter than the double of the fourth; but, if the firſt be greater the ſecond, the double of the firſt is greater than the double e ſecond; wherefore alſo the double of the third is greater the double of the fourth; therefore the third is greater than fourth: In like manner, if the firſt be equal to the ſecond, eſs than it, the third can be proved to be equal to the fourth, eſs than it. Therefore, if the firſt, &c. Q. E. D.

PROP. B. THEOR.

four magnitudes are proportionals, they are pro- portionals alſo when taken inverſely. See N.

the magnitude A be to B, as C is to D, then alſo inverſely to A, as D to C.

ke of B and D any equimultiples ever E and F; and of A and C any e- ultiples whatever G and H. Firſt, Let greater than G, then G is leſs than E; becauſe A is to B, as C is to D, and and C the firſt and third, G and H quimultiples; and of B and D, the ſe- and fourth, E and F are equimulti- and that G is leſs than E, H is alſo than F; that is, F is greater than H; refore E be greater than G, F is great- an H: In like manner, if E be equal F may be ſhewn to be equal to H; if leſs, leſs; and E, F are any equi- ples whatever of B and D, and G, H whatever of A and C; therefore as B



[a] 5. def. 5

is

Book V. is to A, ſo is D to C. If then four magnitudes, &c. E D.

PROP. C. THEOR.

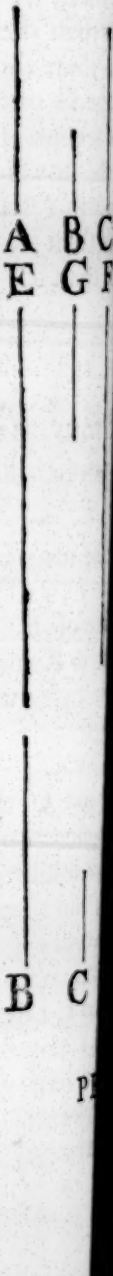
See N. IF the firſt be the ſame multiple of the ſecond, or ſame part of it, that the third is of the four the firſt is to the ſecond, as the third is to the four

Let the firſt A be the ſame multiple of B the ſecond, C the third is of the fourth D: A is to B, as C is to D.

Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H: Then, becauſe A is the ſame multiple of B that C is of D; and that E is the ſame multiple of A, that F is of C; E is the ſame multiple of B, that F is of D [a]; therefore E and F are the ſame multiples of B and D: But G and H are equimultiples of B and D; therefore, if E be a greater multiple of B, than G is, F is a greater multiple of D, than H is of D; that is, if E be greater than G, F is greater than H: In like manner, if E be equal to G, or leſs; F is equal to H, or leſs than it. But E, F are equimultiples, any whatever, of A, C, and G, H any equimultiples whatever of B, D. Therefore A is to B, as C is to D [b].

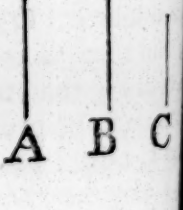
[a] 3. 5.

[b] 5. Def. 5.



Next, Let the firſt A be the ſame part of the ſecond B, that the third C is of the fourth D: A is to B, as C is to D: For B is the ſame multiple of A, that D is of C; wherefore, by the preceeding caſe, B is to A, as D is to C; and inverſely [c] A is to B, as C is to D. Therefore, if the firſt be the ſame multiple, &c. Q. E. D.

[c] B. 5.

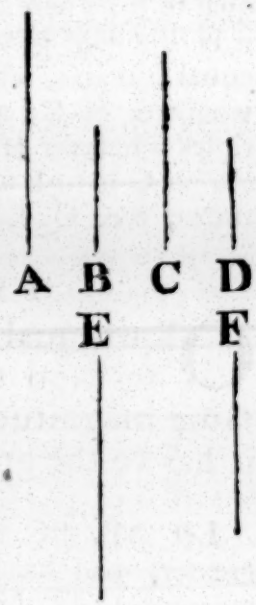


PROP. D. THEOR.

F the first be to the second as the third to the fourth, and if the first be a multiple, or part of e second; the third is the same multiple, or the me part of the fourth. See N.

Let A be to B, as C is to D; and first let A be a multiple B; C is the same multiple of D.

Take E equal to A, and whatever multe A or E is of B, make F the same multe of D: Then, because A is to B, as C is D; and of B the second and D the fourth imultiples have been taken E and F; is to E, as C to F [a]: But A is equal E, therefore C is equal to F [b]: And F he same multiple of D, that A is of B. herefore C is the same multiple of D, t A is of B.



[a] Cor. 4. 5.
[b] A. 5.

Next, Let the first A be a part of the sed B; C the third is the same part of the th D.

See the figure at the foot of the preceeding page.

Because A is to B, as C is to D; then, erfely B is [c] to A, as D to C: But A is a t of B, therefore B is a multiple of A; , by the preceeding case, D is the same ltiple of C; that is, C is the same part of D, that A is of B: erefore, if the first, &c. Q. E. D.

[c] B. 5.

PROP. VII. THEOR.

QUAL magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

et A and B be equal magnitudes, and C any other. A and ave each of them the same ratio to C, and C has the same to each of the magnitudes A and B.

ake of A and B any equimultiples whatever D and E, and

of

Book V.

of C any multiple whatever F: Then, becauſe D is the ſa multiple of A, that E is of B, and that A is [a] 1. Ax. 5. equal to B; D is [a] equal to E: Therefore, if D be greater than F, E is greater than F; and if equal, equal; if leſs, leſs: And D, E are any equimultiples of A, B, and F is any mul- [b] 5. Def. 5. tiple of C. Therefore [b] as A is to C, ſo is B to C.

Likewiſe C has the ſame ratio to A that it has to B: For, having made the ſame conſtruction, D may in like manner be ſhewn equal to E: Therefore, if F be greater than D, it is likewiſe greater than E; and if equal, equal; if leſs, leſs: And F is any multiple whatever of C, and D, E are any equimultiples whatever of A, B. Therefore C is to A, as C is to B [b]. Therefore equal magnitudes, &c. Q. E. D.

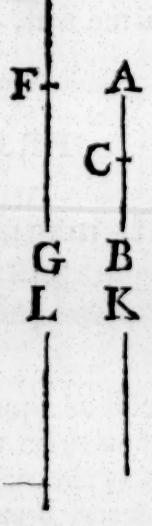
PROP. VIII. THEOR.

See N.

OF unequal magnitudes, the greater has a gre ratio to the ſame than the leſs has: And ſame magnitude has a greater ratio to the leſs t it has to the greater.

Let AB, BC be unequal magnitudes, of which AB i greater, and let D be any magnitude whatever: AB has a greater ratio to D than BC to D: And D has a greater ratio to BC than unto AB.

Fig. 1.



If the magnitude which is not the greater of the two AC, CB, be not leſs than D, take EF, FG the doubles of AC, CB, as in Fig. 1. But, if that which is not the greater of the two AC, CB be leſs than D (as in Fig. 2. and 3.) this magnitude can be multiplied ſo as to become greater than D, whether it be AC or CB. Let it be multiplied until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the ſame multiple of CB: Therefore EF and FG are each of them greater than

Book V.

And in every one of the cases take H the double of D, K triple, and so on, till the multiple of D be that which first comes greater than FG: Let L be that multiple of D which is first greater than FG, and K the multiple of D which is next less than L.

Then, because L is the multiple of D which is the first that comes greater than FG, the next preceeding multiple K is not greater than FG; that is, FG is not less than K: And since EF is the same multiple of AC, that FG is of CB; FG is the same multiple of CB, that EG is of AB [a]; wherefore EG and FG are equimultiples of AB and CB: and it was shewn, that FG was not less than K,

[a] 1. 5.

Fig. 2.

Fig. 3.

but EG was not less than K, by the construction, EF is greater than AC; therefore the whole FG is greater than K and CB together: But K together with D is equal to L; therefore EG is greater than L; but FG is not greater than L; and EG, FG are equimultiples of AB, BC, and L is a multiple of AB; therefore [b] AB has to D a greater ratio than BC has to D.

[b] 7. Def. 5.

Also D has to BC a greater ratio than it has to AB: For, having made the same construction, it may be shewn, in like manner, that L is greater than EG, but that it is not greater than EG: And L is a multiple of AB; and FG, EG are equimultiples of CB, AB; therefore D has to CB a greater ratio [b] than it has to AB. Wherefore, of unequal magnitudes, &c. Q. E. D.

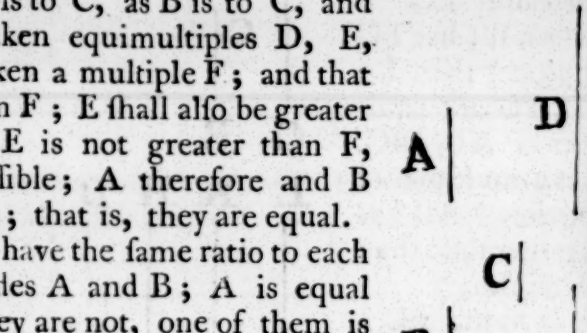
PROP.

PROP. IX. THEOR.

See N. MAGNITUDES which have the ſame ratio to the ſame magnitude are equal to one another: and thoſe to which the ſame magnitude has the ſame ratio are equal to one another.

Let A, B have each of them the ſame ratio to C; A is equal to B: For, if they are not equal, one of them is greater than the other; let A be the greater; then, by what was ſhewn in the preceeding Propoſition, there are ſome equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let ſuch multiples be taken, and let D, E, be the equimultiples of A, B, and F the multiple of C, ſo that D may be greater than F, and E not greater than F. But, becauſe A is to C, as B is to C, and of A, B are taken equimultiples D, E, and of C is taken a multiple F; and that D is greater than F; E ſhall alſo be greater than F [a]; but E is not greater than F, which is impoſſible; A therefore and B are not unequal; that is, they are equal.

[a] 5. Def. 5.



Next, Let C have the ſame ratio to each of the magnitudes A and B; A is equal to B: For, if they are not, one of them is greater than the other; let A be the greater; therefore, as was ſhewn in Prop. 8th, there is ſome multiple F of C, and ſome equimultiples E and D of B and A ſuch, that F is greater than E, and not greater than D; becauſe C is to B, as C is to A, and that F the multiple of the third is greater than E the multiple of the ſecond; F the multiple of the third is greater than D the multiple of the fourth [a]: But F is not greater than D, which is impoſſible. Therefore A is equal to B. Wherefore magnitudes which, &c. Q. E. D.

PROP.

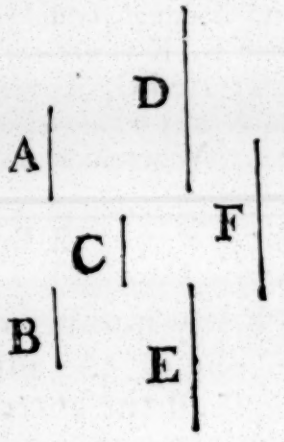
Book v.

PROP. X. THEOR.

THAT magnitude which has a greater ratio than another has unto the ſame magnitude is the ater of the two: And that magnitude to which ſame has a greater ratio than it has unto another gnitude is the leſſer of the two. See N.

et A have to C a greater ratio than B has to C; A is er than B: For, becauſe A has a greater ratio to C, than s to C, there are [a] ſome equimultiples of A and B, and multiple of C ſuch, that the multiple of A is greater than multiple of C, but the multiple of B is not greater than it: hem be taken, and let D, E be equi- iples of A, B, and F a multiple of C that D is greater than F, but E is greater than F: Therefore D is greater E: And, becauſe D and E are equi- iples of A and B, and D is greater E; therefore A is [b] greater than B. ext, Let C have a greater ratio to B it has to A; B is leſs than A: For [a] is ſome multiple F of C, and ſome e- ultiples E and D of B and A ſuch, F is greater than E, but is not greater D: E therefore is leſs than D; and uſe E and D are equimultiples of B A, therefore B is [b] leſs than A. That nitude, therefore, &c. Q. E. D.

[a] 7. Def. 5.



[b] 4. Ax. 5.

PROP. XI. THEOR.

ATIOS that are the ſame to the ſame ratio, are the ſame to one another.

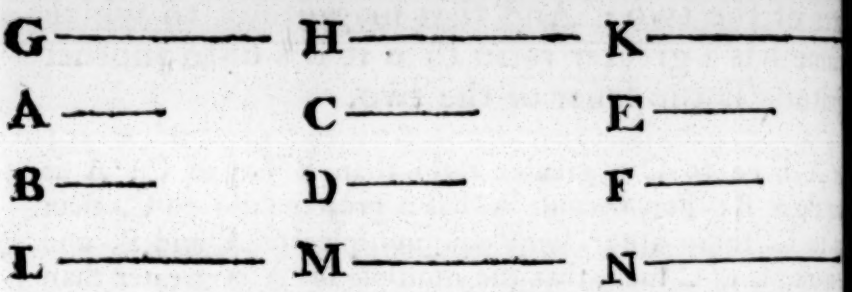
et A be to B, as C is to D; and as C to D, ſo let E be to A is to B, as E to F.

ake of A, C, E, any equimultiples whatever G, H, K; and , D, F any equimultiples whatever , M, N. Therefore, A is to B, as C to D, and G, H are taken equimultiples of

A,

Book V. A, C, and L, M of B, D; if G be greater than L, H is gre
than M; and if equal, equal; and if leſs, leſs [a]. Again,
cauſe C is to D, as E is to F, and H, K are taken equimulti
of C, E; and M, N of D, F; if H be greater than M,
greater than N; and if equal, equal; and if leſs, leſs: But, i

[a] 5. Def. 5.



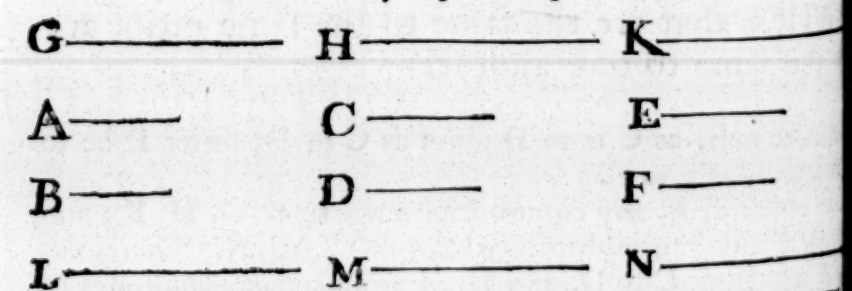
be greater than L, it has been ſhewn, that H is greater than
and if equal, equal; and if leſs, leſs; therefore, if G be gre
than L, K is greater than N; and if equal, equal; and if
leſs: And G, K, are any equimultiples whatever of A, E;
L, N any whatever of B, F: Therefore, as A is to B, ſo is
F [a]. Wherefore ratios that, &c. Q. E. D.

PROP. XII. THEOR.

IF any number of magnitudes be proportionals, one of the antecedents is to its conſequent, ſhall all the antecedents taken together be to all t conſequents.

Let any number of magnitudes A, B, C, D, E, F, be prop tionals; that is, as A is to B, ſo C to D, and E to F: As A to B, ſo ſhall A, C, E together be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H,



and of B, D, F any equimultiples whatever L, M, N: T
becauſe A is to B, as C is to D, and as E to F; and that G,
K

K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; if G be greater than L, H is greater than M, and K greater than N; and if equal, equal; and if leſs, leſs [a]. Wherefore, if G be greater than L, then G, H, K together are greater than L, M, N together; and if equal, equal; and if leſs, leſs. And G, and G, H, K together are any equimultiples of A, and A, C, E together; becauſe, if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the ſame multiple is the whole of the whole [b]: For the ſame reaſon L, and L, M, N are any equimultiples of B, and B, D, F: As therefore A is to B, ſo are A, C, E together to B, D, F together. Wherefore, if any number, &c. Q. E. D. Book V.

[a] 5. def. 5.

[b] 1. 5.

PROP. XIII. THEOR.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth. See N.

Let A the firſt have the ſame ratio to B the ſecond which C the third has to D the fourth, but C the third to D, the fourth, a greater ratio than E the fifth to F the ſixth: Alſo the firſt A ſhall have to the ſecond B a greater ratio than the fifth E to the ſixth F.

Becauſe C has a greater ratio to D, than E to F, there are ſome equimultiples of C and E, and ſome of D and F ſuch, that the multiple of C is greater than the multiple of D, but

M——— G——— H———
A—— C—— E——
B—— D—— F——
N——— K——— L———

the multiple of E is not greater than the multiple of F [a]: Let ſuch be taken, and of C, E let G, H be equimultiples, and K, L equimultiples of D, F, ſo that G be greater than K, but H not greater than L; and whatever multiple G is of C take M the ſame multiple of A; and what multiple K is of D, take N the ſame multiple of B: Then, becauſe A is to B, as C to D, and

[a] 7. def. 5.

Book V. of A and C, M and G are equimultiples: And of B and D, and K are equimultiples; if M be greater than N, G is great[b] 5. def. 5. than K; and if equal, equal; and if leſs, leſs [b]; but G greater than K, therefore M is greater than N: But H is n greater than L; and M, H are equimultiples of A, E; a N, L equimultiples of B, F: Therefore A has a greater ra[a] 7. def. 5. to B, than E has to F[a]. Wherefore, if the firſt, &c. Q. E.

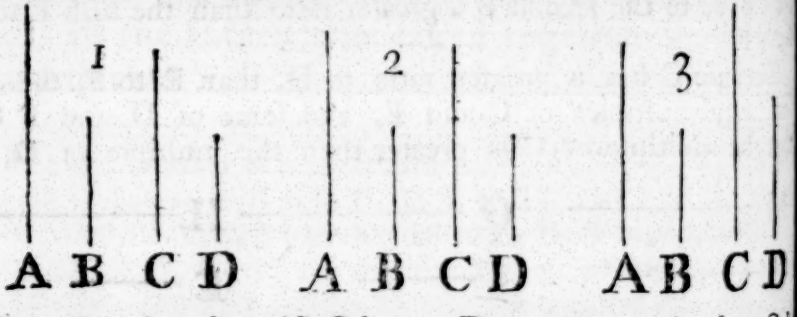
Cor. And if the firſt has a greater ratio to the ſecond, th the third has to the fourth, but the third the ſame ratio to t fourth, which the fifth has to the ſixth: it may be demonſtrat in like manner that the firſt has a greater ratio to the ſecon than the fifth has to the ſixth.

PROP. XIV. THEOR.

See N. IF the firſt has to the ſecond the ſame ratio, whic the third has to the fourth; then, if the firſt b greater than the third, the ſecond ſhall be great than the fourth; and if equal, equal; and if leſs, le

Let the firſt A have to the ſecond B the ſame ratio, which t third C has to the fourth D; if A be greater than C, B greater than D.

Becauſe A is greater than C, and B is any other magnitud[a] 8. 5. A has to B a greater ratio than C to B[a]: But as A is to B,



is C to D; therefore alſo C has to D a greater ratio than C h[b] 13. 5. to B [b]: But of two magnitudes, that to which the ſame has t[c] 10. 5. greater ratio is the leſſer [c]: Wherefore D is leſs than B; th is, B is greater than D.

Secondly, If A be equal to C, B is equal to D: For A is B, as C, that is A, to D; B therefore is equal to D [d].[d] 9. 5.

Thirdly, If A be leſs than C, B ſhall be leſs than D: F C is greater than A, and becauſe C is to D, as A is to B, D greater than B by the firſt caſe; wherefore B is leſs than Therefore, if the firſt, &c. Q. E. D.

PRO

PROP. XV. THEOR.

MAGNITUDES have the ſame ratio to one another which their equimultiples have.

Let AB be the ſame multiple of C, that DE is of F: C is to as AB to DE.

Becauſe AB is the ſame multiple of C that DE is of F; there e as many magnitudes in AB equal to C, there are in DE equal to F: Let AB be ided into magnitudes, each equal to C, . AG, GH, HB; and DE into magnides, each equal to F, viz. DK, KL, LE: hen the number of the firſt AG, GH, HB all be equal to the number of the laſt DK, , LE: And becauſe AG, GH, HB are equal, and that DK, KL, LE are alſo ual to one another; therefore AG is to K, as GH to KL, and as HB to LE [a]: nd as one of the antecedents to its conſeent; ſo are all the antecedents together to the conſequents together [b]; wherefore, as AG is to DK, ſo [b] 12. 5. AB to DE: But AG is equal to C, and DK to F: Therefore as C is to F, ſo is AB to DE. Therefore magnitudes, &c. E. D.

[a] 7. 5.

PROP. XVI. THEOR.

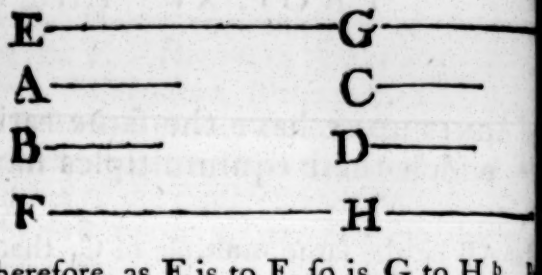
IF four magnitudes of the ſame kind be proportionals, they ſhall alſo be proportionals when taken alternately.

Let the four magnitudes A, B, C, D be proportionals, viz. A to B, ſo C to D: They ſhall alſo be proportionals when taen alternately; that is, A is to C, as B to D

Take of A and B any equimultiples whatever E and F; and C and D take any equimultiples whatever G and H: And

Book V.

becauſe E is the ſame multiple of A, that F is of B, and ſ magnitudes have the ſame ratio to one another which their
[a] 15. 5. quimultiples have [a]; therefore A is to B, as E is to F: But A is to B, ſe is C to D: Wherefore, as C
[b] 11. 5. is to D, ſo [b] is E to F: Again, becauſe G, H are equimultiples of C, D, as C is to D, ſo is G to H [a]; but as C is to D, ſo is E to F. Wherefore, as E is to F, ſo is G to H [b]. B when four magnitudes are proportionals, if the firſt be gre than the third, the ſecond ſhall be greater than the fourth;
[c] 14. 5. if equal, equal; if leſs, leſs [c]. Wherefore, if E be greater t G, F likewiſe is greater than H; and if equal, equal; if leſs, l and E, F are any equimultiples whatever of A, B; and G,
[d] 5. def. 5. any whatever of C, D. Therefore A is to C, as B to D [d]. then four magnitudes, &c. Q. E. D.



PROP. XVII. THEOR.

See N. IF magnitudes taken jointly be proportionals, th ſhall alſo be proportionals when taken ſeparate that is, if two magnitudes together have to one them, the ſame ratio which two others have to o of theſe, the remaining one of the firſt two ſhall ha to the other, the ſame ratio which the remaining o of the laſt two has to the other of theſe.

Let AB, BE, CD, DF be the magnitudes taken jointly wh are proportionals; that is, as AB to BE, ſo is CD to DF; t ſhall alſo be proportionals taken ſeparately, viz. as AE to E ſo CF to FD.

Take of AE, EB, CF, FD any equimultiples whatever G HK, LM, MN; and again, of EB, FD, take any equimulti whatever KX, NP: And becauſe GH is the ſame multiple
[a] 1. 5. AE that HK is of EB, therefore GH is the ſame multiple AE, that GK is of AB: But GH is the ſame multiple of A that LM is of CF; wherefore GK is the ſame multiple of A

Book V.

at LM is of CF. Again, becauſe LM is the ſame multiple of F that MN is of FD; therefore LM is the ſame multiple [a] of F, that LN is of CD: But LM was ſhewn to be the ſame ultiple of CF, that GK is of AB; GK therefore is the ſame ultiple of AB, that LN is of CD; that is, GK, LN are equi-ultiples of AB, CD. Next, becauſe HK is the ſame multiple EB, that MN is of FD; and that KX is o the ſame multiple of EB, that NP is FD; therefore HX is the ſame multiple of EB, that MP is of FD. And becauſe B is to BE, as CD is to DF, and that of B and CD, GK and LN are equimul-ples, and of EB and FD, HX and MP are uimultiples; if GK be greater than HX, en LN is greater than MP; and if equal, ual; and if leſs, leſs [c]: But if GH be reater than KX, by adding the common rt HK to both, GK is greater than HX; herefore alſo LN is greater than MP; nd by taking away MN from both, LM greater than NP: Therefore, if GH be reater than KX, LM is greater than NP. n like manner it may be demonſtrated, at if GH be equal to KX, LM likewiſe is equal to NP; and leſs, leſs: And GH, LM are any equimultiples whatever of E, CF, and KX, NP are any whatever of EB, FD. There-re [c], as AE is to EB, ſo is CF to FD. If then magnitudes, c. Q. E. D.

[a] 1. 5.

[b] 2. 5.

[c] 5. def. 5.

PROP. XVIII. THEOR.

IF magnitudes taken ſeparately be proportionals, they ſhall alſo be proportionals when taken jointly, that s, if the firſt be to the ſecond, as the third to the ourth, the firſt and ſecond together ſhall be to the econd, as the third and fourth together to the fourth.

See N.

Let AE, EB, CF, FD be proportionals; that is, as AE to EB, ſo is CF to FD; they ſhall alſo be proportionals when taken ointly; that is, as AB to BE, ſo CD to DF.

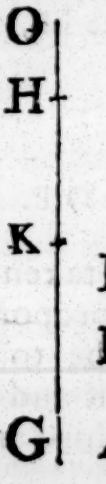
Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN; and again, of BE, DF, take any whatever e-quimultiples KO, NP: And becauſe KO, NP are equimultiples

Book V.

of BE, DF; and that KH, NM are equimultiples likewise of BE, DF, if KO the multiple of BE be greater than KH which is a multiple of the same BE, NP likewise the multiple of DF shall be greater than NM the multiple of the same DF; and if KO be equal to KH, NP shall be equal to NM; and if less, less.

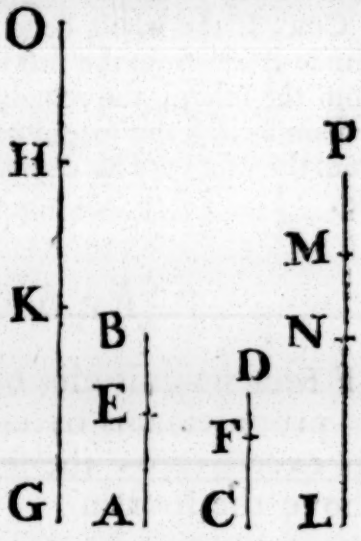
First, Let KO not be greater than KH, therefore NP is not greater than NM: And because GH, HK are equimultiples of AB, BE, and that AB is greater than BE, therefore GH is
[a] 3. Ax. greater [a] than HK; but KO is not greater than KH, wherefore GH is greater than KO. In like manner it may be shewn, that LM is greater than NP. Therefore, if KO be not greater than KH, then GH the multiple of AB is always greater than KO the multiple of BE; and likewise LM the multiple of CD greater than NP the multiple of DF.

Next, Let KO be greater than KH; therefore, as has been shewn, NP is greater than NM: And because the whole GH is the same multiple of the whole AB, that HK is of BE, the remainder GK is the same multiple of
[b] 5. 5. the remainder AE that GH is of AB [b], which is the same that LM is of CD. In like manner, because LM is the same multiple of CD, that MN is of DF, the remainder LN is the same multiple of the remainder CF, that the whole LM is of the whole CD [b]: But it was shewn that LM is the same multiple of CD that GK is of AE; therefore GK is the same multiple of AE that LN is of CF; that is, GK, LN are equimultiples of AE, CF: And because KO, NP are equimultiples of BE, DF, if from KO, NP there be taken KH, NM, which are likewise equimultiples of BE, DF, the remainders HO, MP are either equal to BE,
[c] 6. 5. DF, or equimultiples of them [c]. First, Let HO, MP be equal to BE, DF; and because AE is to EB, as CF to FD, and



t GK, LN are equimultiples of AE, CF; GK ſhall be to Book V.
, as LN to FD [d]: But HO is equal to EB, and MP to FD; [d] Cor. 4. 5.
erefore GK is to HO, as LN to MP. If therefore GK be ater than HO, LN is greater than MP; and if equal, equal; if leſs [e], leſs. [e] Ax. 5.

But let HO, MP be equimultiples of EB, FD; and becauſe is to EB, as CF to FD, and that of AE, CF are taken emultiples GK, LN; and of EB, FD, the equimultiples HO, ; if GK be greater than HO, LN greater than MP; and if equal, eual; and if leſs, leſs [f]; which was ewiſe ſhewn in the preceeding caſe. herefore GH be greater than KO, ing KH from both, GK is greater n HO; wherefore alſo LN is ater than MP; and conſequently, ing NM to both, LM is greater n NP: Therefore, if GH be ater than KO, LM is greater than . In like manner it may be ſhewn, t if GH be equal to KO, LM is ual to NP; and if leſs, leſs. And the caſe in which KO is not greatthan KH, it has been ſhewn that H is always greater than KO, and likewiſe LM than NP: t GH, LM are any equimultiples of AB, CD, and KO, are any whatever of BE, DF; therefore [f] as AB is to BE, is CD to DF. If then magnitudes, &c. Q. E. D.



[f] 5. def. 5.

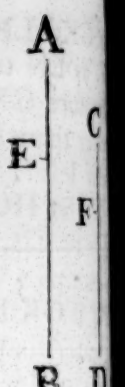
PROP. XIX. THEOR.

F a whole magnitude be to a whole, as a magnitude taken from the firſt is to a magnitude taken from e other; the remainder ſhall be to the remainder the whole to the whole. See N.

Let the whole AB be to the whole CD, as AE a magnitude ken from AB to CF a magnitude taken from CD; the reainder EB ſhall be to the remainder FD, as the whole AB, to e whole CD.

Becauſe AB is to CD, as AE to CF; likewiſe, alternately [a], [a] 16. 5.

Book V. BA is to AE, as DC to CF: And becauſe, if magnitudes taken jointly be proportionals, they are
[b] 17. 5. alſo proportionals [b] when taken ſeparately; therefore as BE is to EA, ſo is DF to FC; and alternately, as BE is to DF, ſo is EA to FC: But as AE to CF, ſo, by the hypotheſis, is AB to CD; therefore alſo BE the remainder ſhall be to the remainder DF, as the whole AB to the whole CD: Wherefore, if the whole, &c. Q. E. D.



Cor. If the whole be to the whole, as a magnitude taken from the firſt is to a magnitude taken from the other; the remainder likewiſe is to the remainder, as the magnitude taken from the firſt to that taken from the other: The demonſtration is contained in the preceeding.

PROP. E. THEOR.

IF four magnitudes be proportionals, they are alſo proportionals by converſion, that is, the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth.

Let AB be to BE, as CD to DF; then BA is to AE, as DC to CF.

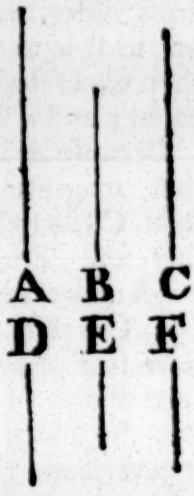
Becauſe AB is to BE, as CD to DF, by diviſion [a], AE is to EB, as CF to FD; and by inverſion [b], BE is to EA, as DF to FC. Wherefore, by compoſition [c]: BA is to AE, as DC is to CF: If, therefore, four, &c. Q. E. D.

[a] 17. 5.
[b] B. 5.
[c] 18. 5.

PROP. XX. THEOR.

See N. IF there be three magnitudes, and other three which taken two and two have the ſame ratio; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs,

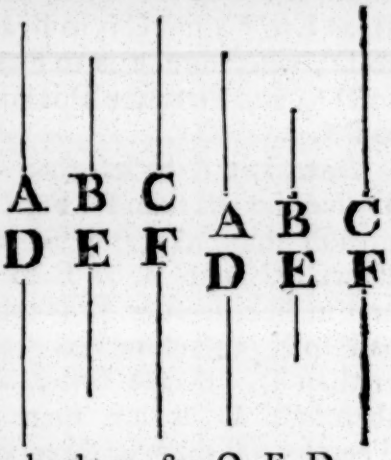
Let A, B, C be three magnitudes, and D, E, F other three, which taken two and two have the ſame ratio, viz. as A is to B, ſo is D to E; and as B to C, ſo is E to F. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſs. Book V.



Becauſe A is greater than C, and B is any other magnitude, and that the greater has to the ſame magnitude a greater ratio than the leſs has to it [a]; therefore A has to B a greater ratio than C has to B: But as D is to E, ſo is A to B; therefore [b] D has to E a greater ratio than C to B: And becauſe B is to C, as E to F, by inverſion, C is to B, as F is to E; and D was ſhewn to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E [c]: But the magnitude which has a greater ratio than another to the ſame magnitude, is the greater of the two [d]: D is therefore greater than F.

[a] 8. 5.
[b] 13. 5.
[c] Cor. 13. 5.
[d] 10. 5.

Secondly, Let A be equal to C; D ſhall be equal to F: Becauſe A and C are equal to one another, A is to B, as C is to B [e]: but A is to B, as D to E; and C is to B, as F to E; wherefore D is to E, as F to E [f]; and therefore D is equal to F [g].



[e] 7. 5.
[f] 11. 5.
[g] 9. 5.

Next, Let A be leſs than C; D ſhall be leſs than F: For C is greater than A, and, as was ſhewn in the firſt caſe, C is to B, as F to E, and in like manner B is to A, as E to D; therefore F is greater than D, by the firſt caſe; and therefore D is leſs than F. Therefore, if there be three, &c. Q. E. D.

PROP. XXI. THEOR.

IF there be three magnitudes, and other three, which have the ſame ratio taken two and two, but in a croſs order; if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

Let

Book V.

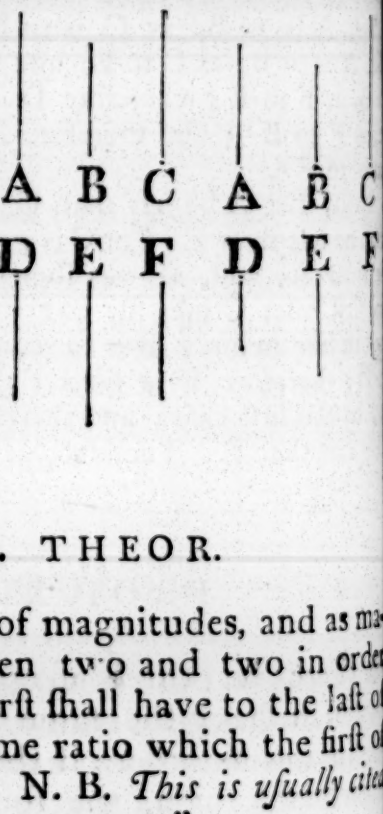
Let A, B, C be three magnitudes, and D, E, F, other three, which have the ſame ratio taken two and two, but in croſs order, viz. as A is to B, ſo is E to F, and as B is to C, ſo is D to E. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſs

Becauſe A is greater than C, and B is any other magnitude, A has to B a greater ratio [a] than C has to B: But as E to F, ſo is A to B; therefore [b] E has to F a greater ratio than C to B: And becauſe B is to C, as D to E, by inverſion, C is to B, as E to D: And E was ſhewn to have to F a greater ratio than C to B; therefore [c] E has to F a greater ratio than E to D [c]; but the magnitude to which the ſame has a greater ratio than it has to another, is the leſſer of the two [d]; F therefore is leſs than D; that is, D is greater than F.

[a] 8. 5. [b] 13. 5. [c] Cor. 13. 5. [d] 10. 5.

Secondly, Let A be equal to C; D ſhall be equal to F. Becauſe A and C are equal, A is [e] to B, as C is to B: But A is to B, as E to F; and C is to B, as E to D; wherefore [f] is to F as E to D [f]; and therefore D is equal to F [g].

[e] 7. 5. [f] 11. 5. [g] 9. 5.



Next, Let A be leſs than C; D ſhall be leſs than F: For C is greater than A, and, as was ſhewn, C is to B, as E to D, and in like manner B is to A, as F to E; therefore F is greater than D, by caſe firſt; and therefore D is leſs than F. Therefore, if there be three, &c. Q. E. D.

PROP. XXII. THEOR.

See N.

IF there be any number of magnitudes, and as many others, which taken two and two in order have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt. N. B. *This is uſually cited by the words "ex aequali," or, "ex aequo."*

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First, Let there be three magnitudes A, B, C, and as many others D, E, F, which taken two and two have the same ratio, that is, such that A is to B, as D to E; and as B is to C, so is E to F; A shall be to C, as D to F.

Take of A and D any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N: Then because A is to B, as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; G is to K, so is [a] H to L: For the same reason K is to M, as L to N: And because there are three magnitudes G, K, M, and other three H, L, N, which two and two have the same ratio; if G be greater than M, H is greater than N; and if equal, equal; and if less, less [b]: And G, H are any equimultiples whatever of A, D, and M, N are any equimultiples whatever of C, F: Therefore [c] as A is to C, so is D to F.

A B C D E F
G K M H L N

[a] 4. 5.

[b] 20. 5.

[c] 5. Def. 5.

Next, Let there be four magnitudes A, B, C, D, and other four E, F, G, H, which two and two have the same ratio, viz. as A is to B, so is E to F; and as B to C, so F to G; and as C to D, so G to H: A shall be to D, as E to H.

A. B. C. D.
E. F. G. H.

Because A, B, C are three magnitudes, and E, F, G other three, which taken two and two have the same ratio; by the foregoing case, A is to C, as E to G: But C is to D, as G is to H; wherefore again, by the first case, A is to D, as E to H; and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

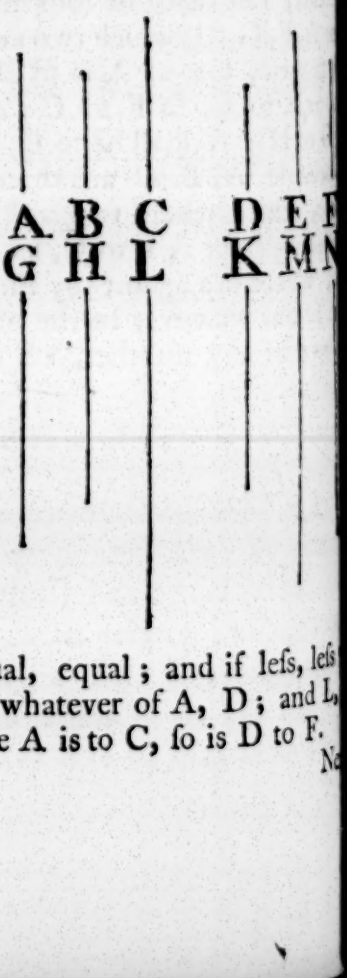
PROP.

PROP. XXIII. THEOR.

See N. IF there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last. N. B. *This is usually cited by the words "ex aequali in proportione perturbata;" or "ex aequo perturbate."*

First, Let there be three magnitudes A, B, C, and other three D, E, F, which taken two and two in a cross order have the same ratio, that is, such that A is to B, as E to F; and as B is to C, so is D to E: A is to C, as D to F.

Take of A, B, D any equimultiples whatever G, H, K; and of C, E, F any equimultiples whatever L, M, N: And because G, H are equimultiples of A, B, and that magnitudes have the same ratio which their equimultiples have [a]; as A is to B, so is G to H: And for the same reason, as E is to F, so is M to N: But as A is to B, so is E to F; as therefore G is to H, so is M to N [b]: And because as B is to C, so is D to E, and that H, K are equimultiples of B, D, and L, M of C, E; as H is to L, so is K to M [c]: And it has been shewn that G is to H, as M to N: Then, because there are three magnitudes G, H, L, and other three K, M, N which have the same ratio taken two and two in a cross order; if G be greater than L, K is greater than N; and if equal, equal; and if less, less [d]: and G, K are any equimultiples whatever of A, D; and L, N any whatever of C, F; as therefore A is to C, so is D to F.



[a] 15. 5.
[b] 11. 5.
[c] 4. 5.
[d] 21. 5.

No

Next, Let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which, taken two and two in a croſs order, have the ſame ratio, viz. A to B, as G to H; B to C, as F to G; and C to D, as E to F: A is to D, as E to H. Book V.

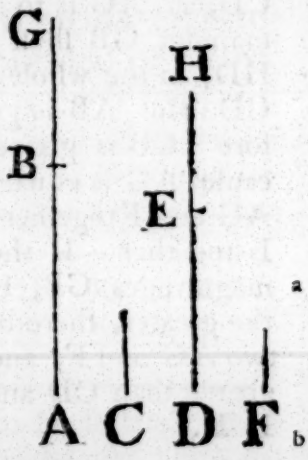
A.	B.	C.	D
E.	F.	G.	H.

Becauſe A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in a croſs order, have the ſame ratio; by the firſt caſe, A is to C, as F to H: But C is to D, as E to F; wherefore again, by the firſt caſe, A is to D as E to H: And ſo on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

PROP. XXIV. THEOR.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth; and the fifth to the ſecond the ſame ratio which the ſixth has to the fourth; the firſt and fifth together ſhall have to the ſecond, the ſame ratio which the third and ſixth together have to the fourth. See N.

Let AB the firſt have to C the ſecond the ſame ratio, which DE the third has to F the fourth; and let BG the fifth have to C the ſecond the ſame ratio, which EH the ſixth has to F the fourth: AG, the firſt and fifth together, ſhall have to C the ſecond the ſame ratio, which DH, the third and ſixth together, has to F the fourth.



Becauſe BG is to C, as EH to F; by inverſion, C is to BG, as F to EH: And becauſe as AB is to C, ſo is DE to F; and as C to BG, ſo F to EH; ex aequali [a], AB is to BG, as DE to EH: And becauſe theſe magnitudes are proportionals, they ſhall likewiſe be proportionals when taken jointly [b]; as therefore AG is to GB, ſo is DH to HE; but as GB to C, ſo is HE to F. Therefore, ex aequali [a], as AG is to C, ſo is DH to F. Wherefore, if the firſt, &c. Q. E. D.

[a] 22. 5.

[b] 18. 5.

Cor. 1. If the ſame hypotheſis be made as in the propoſition, the exceſs of the firſt and fifth ſhall be to the ſecond, as the

Book V. the excefs of the third and fixth to the fourth: The demonſtration of this is the ſame with that of the Propoſition, if diviſion be uſed inſtead of compoſition.

Cor. 2. The propoſition holds true of two ranks of magnitudes, whatever be their number, of which each of the firſt rank has to a ſecond magnitude the ſame ratio that the corresponding one of the ſecond rank has to a fourth magnitude; as is manifeſt.

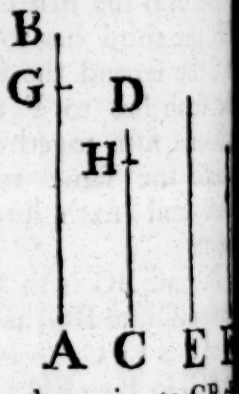
PROP. XXV. THEOR.

IF four magnitudes of the ſame kind are proportionals, the greateſt and leaſt of them together are greater than the other two together.

Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greateſt of them, and conſequently F the leaſt [a]. AB together with F are greater than CD together with E.

[a] A. & 14. 15.

Take AG equal to E, and CH equal to F: Then, becauſe AB is to CD, ſo is E to F, and that AG is equal to E, and CH equal to F; AB is to CD, as AG to CH. And becauſe AB the whole is to the whole CD, as AG is to CH; likewiſe the remainder GB ſhall be to the remainder HD, as the whole AB is to the whole [b] CD: But AB is greater than CD, therefore [c] GB is greater than HD: And becauſe AG is equal to E, and CH to F; AG and F together are equal to CH and E together. If therefore to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and CH and E to HD; AB and F together are greater than CD and E. Therefore, if four magnitudes, &c. Q. E. D.



[b] 19. 5.

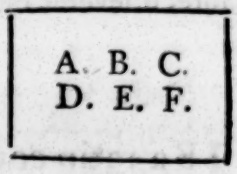
[c] A. 5.

PROP. F. THEOR.

See N.

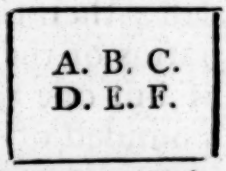
RATIOS which are compounded of the ſame ratios, are the ſame with one another.

Let A be to B, as D to E; and B to C, as E to F: The ra- Book V.
which is compounded of the ratios of
to B, and B to C, which, by the definition
compound ratio, is the ratio of A to C, is
ſame with the ratio of D to F, which, by
ſame definition, is compounded of the ra-
s of D to E, and E to F.



Becauſe there are three magnitudes A, B, C, and three o-
ers D, E, F which taken two and two in order, have the ſame
io; ex aequali, A is to C, as D to F [a]. [a] 22. 5.

Next, Let A be to B, as E to F, and B to C, as D to E;
refore, *ex aequali in proportione perturbata*
A is to C, as D to F; that is, the ratio of A
C, which is compounded of the ratios of A to
and B to C, is the ſame with the ratio of D
F, which is compounded of the ratios of D
E, and E to F: And in like manner the propoſition may be
monſtrated, whatever be the number of ratios in either caſe.

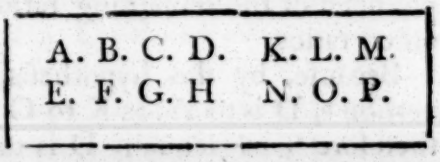


[b] 23. 5.

PROP. G. THEOR.

IF ſeveral ratios be the ſame with ſeveral ratios, See N.
each to each; the ratio which is compounded of
tios which are the ſame with the firſt ratios, each to
ch, is the ſame with the ratio compounded of ratios
hich are the ſame with the other ratios, each to each.

Let A be to B, as E to F; and C to D, as G to H: And let
be to B, as K to L; and C to D, as L to M: Then the ra-
of K to M, by the definition
compound ratio, is compound-
of the ratios of K to L, and
to M, which are the ſame with
ratios of A to B, and C to D:
nd as E to F, ſo let N be to O; and as G to H, ſo let O be to
then the ratio of N to P is compounded of the ratios of N to
and O to P, which are the ſame with the ratios of E to F,
d G to H: And it is to be ſhewn that the ratio of K to M,
the ſame with the ratio of N to P, or that K is to M, as N to P.
Becauſe K is to L, as (A to B, that is, as E to F, that is, as)
to O; and as L to M, ſo is (C to D, and ſo is G to H,
and



Book V. and ſo is) O to P: Ex aequali [a], K is to M, as N to P. Therefore, if ſeveral ratios, &c. Q. E. D.

[a] 22. 5.

PROP. H. THEOR.

See N. IF a ratio compounded of ſeveral ratios be the ſame with a ratio compounded of any other ratios, and if one of the firſt ratios, or a ratio compounded of any of the firſt, be the ſame with one of the laſt ratios, or with the ratio compounded of any of the laſt; then the ratio compounded of the remaining ratios of the firſt, or the remaining ratio of the firſt if but one remain, is the ſame with the ratio compounded of thoſe remaining of the laſt, or with the remaining ratio of the laſt.

Let the firſt ratios be thoſe of A to B, B to C, C to D, D to E, and E to F; and let the other ratios be thoſe of G to H, H to K, K to L, and L to M: Alſo let the ratio of A to F, which is compounded of [a] the firſt ratios be the ſame with the ratio of G to M, which is compounded of the other ratios: And beſides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, C to D, be the ſame with the ratio of G to K, which is compounded of the ratios of G to H, and H to K: Then the ratio compounded of the remaining firſt ratios, to wit, of the ratios of D to E, and E to F, which compounded ratio is the ratio of D to F, is the ſame with the ratio of K to M, which is compounded of the remaining ratios of K to L, and L to M of the other ratios.

A. B. C. D. E. F.
G. H. K. L. M.

[a] Definition of compounded ratio.

[b] B. 5.
[c] 22. 5.

Becauſe, by the hypotheſis, A is to D, as G to K, by converſion [b], D is to A, as K to G; and as A is to F, ſo is G to M; therefore [c], ex aequali, D is to F, as K to M. If therefore a ratio which is, &c. Q. E. D.

PRO

PROP. K. THEOR.

IF there be any number of ratios, and any number of other ratios ſuch, that the ratio compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame, each to each, with the laſt ratios; and if one of the firſt ratios, or the ratio which is compounded of ratios which are the ſame with ſeveral of the firſt ratios, each to each, be the ſame with one of the laſt ratios, or with the ratio compounded of ratios which are the ſame, each to each, with ſeveral of the laſt ratios: Then the ratio compounded of ratios which are the ſame with the remaining ratios of the firſt, each to each, or the remaining ratio of the firſt, if but one remain; is the ſame with the ratio compounded of ratios which are the ſame with thoſe remaining of the laſt, each to each, or with the remaining ratio of the laſt. See N.

Let the ratios of A to B, C to D, E to F be the firſt ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios: And let A be to B, as S to T; and C to D, as T to V; and E to F, as V to X: Therefore, by the definition of compound ratio, the ratio of S to X is compounded

		h, k, l.		
A, B;	C, D;	E, F.		S, T, V, X.
G, H; K, L;	M, N;	O, P;	Q, R.	Y, Z, a, b, c, d.
e, f, g.	m, n, o, p.			

of the ratios of S to T, T to V, and V to X, which are the ſame with the ratios of A to B, C to D, E to F, each to each: Alſo as G to H, ſo let Y be to Z; and K to L, as Z to a; M to N, as a to b, O to P, as b to c; and Q to R, as c to d: Therefore, by the ſame definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to c, and

Book V. c to d, which are the ſame, each to each, with the ratios of G to H, K to L, M to N, O to P, and Q to R: Therefore, by the hypotheſis, S is to X, as Y to d: Alſo let the ratio of A to B, that is, the ratio of S to T, which is one of the firſt ratios, be the ſame with the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the hypotheſis, are the ſame with the ratios of G to H, and K to L, two of the other ratios; and let the ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the ſame with the remaining firſt ratios, viz. of C to D, and E to F; alſo let the ratio of m to p be that which is compounded of the ratios of m to n, n to o, and o to p, which are the ſame each to each, with the remaining other ratios, viz. of M to N, O to P, and Q to R: Then the ratio of h to l is the ſame with the ratio of m to p, or h is to l, as m to p.

	h, k, l.	
A, B;	C, D; E, F.	S, T, V, X.
G, H; K, L;	M, N; O, P; Q, R.	Y, Z, a, b, c, d.
e, f, g.	m, n, o, p.	

Becauſe e is to f, as (G to H, that is, as) Y to Z; and f to g, as (K to L, that is, as) Z to a; therefore, ex aequali, e is to g, as Y to a: And, by the hypotheſis, A is to B, that is S to T, as e to g; wherefore S is to T, as Y to a; and, by inverſion, T is to S, as a to Y; and S is to X, as Y to d; therefore, ex aequali, T is to X, as a to d: Alſo, becauſe h is to k, as (C to D, that is, as) T to V; and k is to l, as (E to F, that is, as) V to X; therefore, ex aequali, h is to l, as T to X: In like manner, it may be demonſtrated, that m is to p, as a to d: And it was ſhewn, that T is to X, as a to d: Therefore
[a] 11. 5. [a] h is to l, as m to p. Q. E. D.

The propoſitions G and K are uſually, for the ſake of brevity, expreſſed in the ſame terms with propoſitions F and H: And therefore it was proper to ſhew the true meaning of them when they are ſo expreſſed; eſpecially ſince they are very frequently made uſe of by geometers.

THE

THE ELEMENTS OF EUCLID.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are thoſe which have their ſeveral angles equal, each to each, and the ſides about the equal angles proportionals.

II.

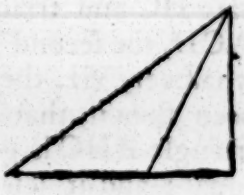
Reciprocal figures, viz. triangles and parallelograms, are ſuch as have their ſides about two of their angles proportionals in ſuch manner, that a ſide of the firſt figure is to a ſide of the other, as the remaining ſide of this other is to the remaining ſide of the firſt." See N.

III.

A ſtraight line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

The altitude of any figure is the ſtraight line drawn from its vertex perpendicular to the baſe.

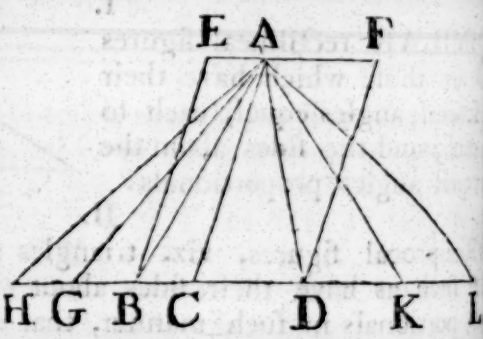


PROP. I. THEOR.

See N. TRIANGLES and parallelograms of the ſame altitude are one to another as their baſes.

Let the triangles ABC, ACD, and the parallelograms EC CF have the ſame altitude, viz. the perpendicular drawn from the point A to BD: Then, as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, L, and take any number of ſtraight lines BG, GH, each equal to the baſe BC; and DK, KL, any number of them, each equal to the baſe CD; and join AG, AH, AK, AL: Then, becauſe CB, BG, GH are all equal, the triangles AHG, AGB, ABC
[a] 38. 1. are all equal [a]: Therefore, whatever multiple the baſe HC is of the baſe BC, the ſame multiple is the triangle AHC of the triangle ABC: For the ſame reaſon, whatever multiple the baſe LC is of the baſe CD, the ſame multiple is the triangle ALC of the triangle ADC: And if the baſe HC be equal to the baſe CL, the triangle AHC is alſo equal to the triangle ALC [a]; and if the baſe HC be greater than the baſe CL, likewiſe the triangle AHC is greater than the triangle ALC; and if leſs, leſs: Therefore, ſince there are four magnitudes, viz. the two baſes BC, CD, and the two triangles ABC, ACD; and of the baſe BC and the triangle ABC, the firſt and third, any equimultiples whatever have been taken, viz. the baſe HC and triangle AHC; and of the baſe CD and triangle ACD, the ſecond and fourth, have been taken any equimultiples whatever, viz. the baſe CL and triangle ALC; and that it has been ſhewn, that if the baſe HC be greater than the baſe CL, the triangle AHC is greater than the triangle ALC; and if equal; and if leſs, leſs: Therefore [b] as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD.



[b] 5. Def. 5.

And becauſe the parallelogram CE is double of the triangle ABC

ABC [c], and the parallelogram CF double of the triangle ACD, Book VI.
nd that magnitudes have the ſame ratio which their equimul- [c] 41. 1.
ples have [d]; as the triangle ABC is to the triangle ACD, ſo [d] 15. 5.
the parallelogram EC to the parallelogram CF: And becauſe
has been ſhewn, that as the baſe BC is to the baſe CD, ſo is
e triangle ABC to the triangle ACD; and as the triangle
ABC to the triangle ACD, ſo is the parallelogram EC to the
rallelogram CF; therefore as the baſe BC is to the baſe CD,
is [e] the parallelogram EC to the parallelogram CF. Where- [e] 11. 5.
re triangles, &c. Q. E. D.

Cor. From this it is plain, that triangles and parallelograms
at have equal altitudes, are one to another as their baſes.

Let the figures be placed ſo as to have their baſes in the ſame
raight line; and having drawn perpendiculars from the vertices
the triangles to the baſes, the ſtraight line which joins the
rtices is parallel to that in which their baſes are [f], becauſe the [f] 33. 1.
rpendiculars are both equal and parallel to one another: Then,
the ſame conſtruction be made as in the propoſition, the de-
onſtration will be the ſame.

PROP. II. THEOR.

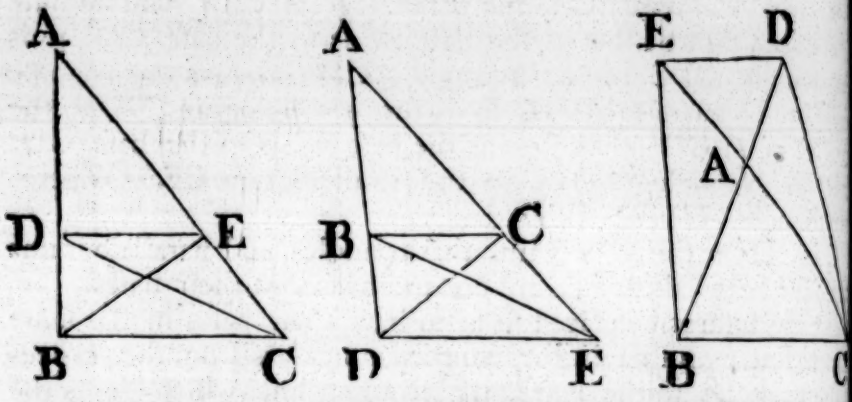
IF a ſtraight line be drawn parallel to one of the See N.
ſides of a triangle, it ſhall cut the other ſides, or
theſe produced, proportionally: And if the ſides, or
the ſides produced, be cut proportionally, the
traight line which joins the points of ſection
hall be parallel to the remaining ſide of the triangle.

Let DE be drawn parallel to BC one of the ſides of the tri-
ngle ABC: BD is to DA, as CE to EA.

Join BE, CD; then the triangle BDE is equal to the tri-
ngle CDE [a], becauſe they are on the ſame baſe DE, and be- [a] 37. 1.
ween the ſame parallels DE, BC: ADE is another triangle,
nd equal magnitudes have to the ſame, the ſame ratio [b]; there- [b] 7. 5.
ore as the triangle BDE to the triangle ADE, ſo is the tri-
ngle CDE to the triangle ADE; but as the triangle BDE to
he triangle ADE, ſo is [c] BD to DA, becauſe having the ſame [c] 1. 6.
ltitude, viz. the perpendicular drawn from the point E to
AB, they are to one another as their baſes; and for the ſame

Book VI. reaſon, as the triangle CDE to the triangle ADE, ſo is CE
EA. Therefore as BD to DA ; ſo is CE to EA [d].
[d] 11. 5.

Next, Let the ſides AB, AC of the triangle ABC, or the



produced, be cut proportionally in the points D, E, that is, ſ
that BD be to DA, as CE to EA, and join DE: DE is para
lel to BC.

The ſame conſtruction being made, becauſe as BD to DA
ſo is CE to EA; and as BD to DA, ſo is the triangle BDE t
[e] 1. 6. the triangle ADE [e]; and as CE to EA, ſo is the triangle CDE
to the triangle ADE; therefore the triangle BDE is to th
triangle ADE, as the triangle CDE to the triangle ADE
that is, the triangles BDE, CDE have the ſame ratio to th
[f] 9. 5. triangle ADE; and therefore [f] the triangle BDE is equal t
the triangle CDE: And they are on the ſame baſe DE; but
[g] 39. 1. qual triangles on the ſame baſe are between the ſame parallels
therefore DE is parallel to BC. Wherefore, if a ſtraight lin
&c. Q. E. D.

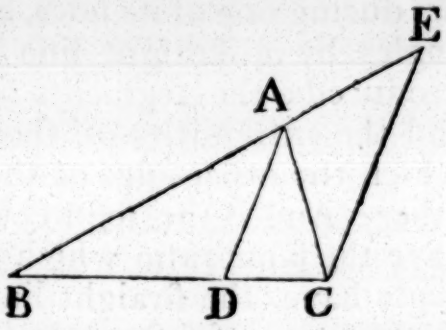
PROP. III. THEOR.

IF the angle of a triangle be divided into two equa
angles, by a ſtraight line which alſo cuts the baſe
the ſegments of the baſe ſhall have the ſame rati
which the other ſides of the triangle have to one an
other: And if the ſegments of the baſe hav
the ſame ratio which the other ſides of the triang
have to one another, the ſtraight line drawn fro
the vertex to the point of ſection, divides the ve
tical angle into two equal angles.

Let the angle BAC of any triangle ABC be divided into tw
equal angles by the ſtraight line AD: BD is to DC, as BA to AC

Thr

Thro' the point C draw CE parallel [a] to DA, and let BA produced meet CE in E. Becauſe the ſtraight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD [b]: But CAD, by the hypotheſis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, becauſe the ſtraight line BAE meets the parallels AD, EC, the outward angle BAD is equal to the inward and oppoſite angle AEC: But the angle ACE has been proved equal to the angle BAD; therefore alſo ACE is equal to the angle AEC, and conſequently the ſide AE is equal to the ſide [c] AC: And becauſe AD is drawn parallel to one of the ſides of the triangle BCE, viz. to EC, BD is to DC, as BA to AE [d]; but AE is equal to AC; therefore as BD to DC, ſo is BA to AC [e].



Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the ſtraight line AD.

The ſame conſtruction being made; becauſe as BD to DC, ſo is BA to AC; and as BD to DC, ſo is BA to AE [d], becauſe AD is parallel to EC; therefore BA is to AC, as BA to AE [f]: conſequently AC is equal to AE [g], and the angle AEC is therefore equal to the angle ACE [h]: But the angle AEC is equal to the outward and oppoſite angle BAD; and the angle ACE is equal to the alternate angle CAD [b]: Wherefore alſo the angle BAD is equal to the angle CAD: Therefore the angle BAC is cut into two equal angles by the ſtraight line AD. Therefore, if the angle, &c. Q. E. D.

Book VI.

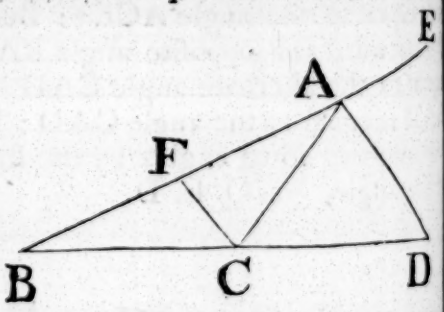
[a] 31. 1.
[b] 29. 1.
[c] 6. 1.
[d] 2. 6.
[e] 7. 5.
[f] 11. 5.
[g] 9. 5.
[h] 5. 1.

PROP. A. THEOR.

IF the outward angle of a triangle made by producing one of its ſides, be divided into two equal angles, by a ſtraight line which alſo cuts the baſe produced; the ſegments between the dividing line and the extremities of the baſe have the ſame ratio which the other ſides of the triangle have to one another: And if the ſegments of the baſe produced have the ſame ratio which the other ſides of the triangle have, the ſtraight line drawn from the vertex to the point of ſection divides the outward angle of the triangle into two equal angles.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the ſtraight line AD which meets the baſe produced in D: BD is to DC, as BA to AC.

[a] 31. 1. Through C draw CF parallel to AD [a]; and becauſe the ſtraight line AC meets the parallels AD, FC, the angle ACF is equal [b] 29. 1. to the alternate angle CAD [b]: But CAD is equal to the angle [c] Hyp. DAE [c]; therefore alſo DAE is equal to the angle ACF. Again, becauſe the ſtraight line FAE meets the parallels AD, FC, the outward angle DAE is equal to the inward and oppoſite angle CFA: But the angle ACF has been proved equal to the angle DAE; therefore alſo the angle ACF is equal to the angle CFA, and conſequently the ſide AF is equal to the ſide AC [d]: And becauſe AD is parallel to FC a ſide of the triangle [d] 6. 1. BCF, BD is to DC, as BA to AF [e]; but AF is equal to AC; [e] 2. 6. as therefore BD is to DC, ſo is BA to AC.



Let now BD be to DC, as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

The ſame conſtruction being made, becauſe BD is to DC, as BA to AC; and that BD is alſo to DC, as BA to AF [e]; [f] 11. 5. therefore BA is to AC, as BA to AF [f] wherefore AC is equal [g] 9. 5. to AF [g], and the angle AFC equal [h] to the angle ACF: But [h] 5. 1. the

angle AFC is equal to the outward angle EAD, and the gle ACF to the alternate angle CAD; therefore alſo EAD is al to the angle CAD. Wherefore, if the outward, &c. Q. D.

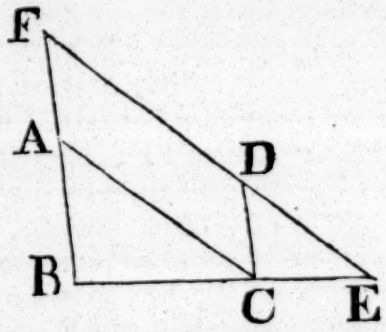
PROP. IV. THEOR.

THE ſides about the equal angles of equiangular triangles are proportionals; and thoſe which oppoſite to the equal angles are homologous ſides, at is, are the antecedents or conſequents of the tios.

Let ABC, DCE be equiangular triangles, having the angle BC equal to the angle DCE, and the angle ACB to the angle EC, and conſequently [a] the angle BAC equal to the angle DE. The ſides about the equal angles of the triangles ABC, CE are proportionals; and thoſe are the homologous ſides ich are oppoſite to the equal angles.

[a] 32. 1.

Let the triangle DCE be placed ſo that its ſide CE may be tiguous to BC, and in the ſame ſtraight line with it: And auſe the angles ABC, ACB are together leſs than two right gles [b]; ABC and DEC, which is al to ACB, are alſo leſs than o right angles, wherefore BA, D produced ſhall meet [c]; let them produced and meet in the point And becauſe the angle ABC is al to the angle DCE, BF is pa- el [d] to CD. Again, becauſe the le ACB is equal to the angle C, AC is parallel to FE [d]: erefore FACD is a parallelogram; and conſequently AF is al to CD, and AC to FD [e]: And becauſe AC is parallel to one of the ſides of the triangle FBE, BA is to AF, as BC CE [f]: But AF is equal to CD; therefore [g] as BA to CD, ſo BC to CE; and alternately, as AB to BC, ſo is DC to CE [b]: ain, becauſe CD is parallel to BF, as BC to CE, ſo is FD to [f]; but FD is equal to AC; therefore as BC to CE, ſo is AC DE: And alternately, as BC to CA, ſo CE to ED: Therefore, auſe it has been proved that AB is to BC, as DC to CE; as BC to CA, ſo CE to ED, ex aequali [h], BA is to AC, as to DE. Therefore the ſides, &c. Q. E. D.



[b] 17. 1.

[c] 12. Ax. 1.

[d] 28. 1.

[e] 34. 1.

[f] 2. 6.

[g] 7. 5.

[h] 22. 5.

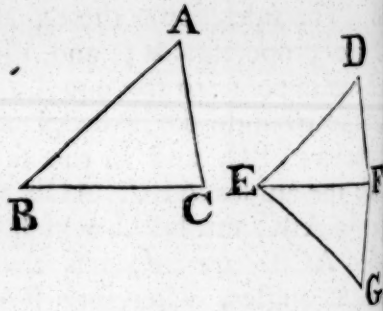
PROB.

PROP. V. THEOR.

IF the ſides of two triangles, about each of the angles, be proportionals, the triangles ſhall be equiangular, and have their equal angles oppoſite to the homologous ſides.

Let the triangles ABC, DEF have their ſides proportional, ſo that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and conſequently, ex aequali, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and the equal angles are oppoſite to the homologous ſides, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and also BAC to EDF.

[a] 23. 1. At the points E, F in the ſtraight line EF make [a] the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle BAC is equal to the remaining angle



[b] 32. 1. EGF [b], and the triangle ABC is therefore equiangular to the triangle GEF; and conſequently they have their ſides oppoſite to the equal angles proportionals [c]: Wherefore, as AB to BC, ſo is GE to EF; but as AB to BC, ſo is DE to EF; therefore as DE

[c] 4. 6.

[d] 11. 5. EF, ſo [d] GE to EF: Therefore DE and GE have the ſame ratio

[e] 9. 5. to EF, and conſequently are equal [e]: For the ſame reaſon DF is equal to FG: And becauſe, in the triangles DEF, GEF, DE is equal to EG, and EF common, the two ſides DE, EF are equal to the two GE, EF, and the baſe DF is equal to the baſe GF;

[f] 8. 1. therefore the angle DEF is equal [f] to the angle GEF, and the other angles to the other angles which are ſubtended by the equal ſides

[g] 4. 1. [g]. Wherefore the angle DFE is equal to the angle GFE, and EDF to EGF: And becauſe the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF: For the ſame reaſon the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if the ſides, &c. Q. E. D.

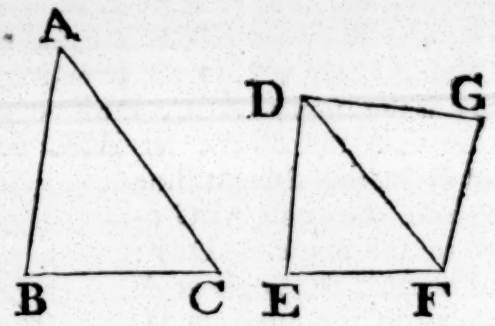
PRO

PROP. VI. THEOR.

F two triangles have one angle of the one equal to one angle of the other, and the ſides about e equal angles proportionals, the triangles ſhall be quiangular, and ſhall have thoſe angles equal which e oppoſite to the homologous ſides.

Let the triangles ABC, DEF have the angle BAC in the e equal to the angle EDF in the other, and the ſides about oſe angles proportionals; that is, BA to AC, as ED to DF: e triangles ABC, DEF are equiangular, and have the angle BC equal to the angle DEF, and ACB to DFE.

At the points D, F, in the ſtraight line DF, make [a] the gle FDG equal to either of the angles BAC, EDF; and the gle DFG equal to the gle ACB: Wherefore remaining angle at B equal to the remaining e at G [b], and conſe-ently the triangle ABC equiangular to the tri-gle DGF; and there-e as BA to AC, ſo is [c] D to DF: But, by the potheſis, as BA to AC, is ED to DF; as therefore ED to DF, ſo is [d] GD to DF; erefore ED is equal [e] to DG; and DF is common to the two ngles EDF, GDF: Therefore the two ſides ED, DF are e-l to the two ſides GD, DF; and the angle EDF is equal to angle GDF; wherefore the baſe EF is equal to the baſe FG [f], the triangle EDF to the triangle GDF, and the remaining les to the remaining angles, each to each, which are ſub-ded by the equal ſides: Therefore the angle DFG is equal to angle DFE, and the angle at G to the angle at E: But the gle DFG is equal to the angle ACB; therefore the angle CB is equal to the angle DFE: And the angle BAC is equal the angle EDF [g]; wherefore alſo the remaining angle at B equal to the remaining angle at E. Therefore the triangle BC is equiangular to the triangle DEF. Wherefore, if two ngles, &c. Q. E. D.



[a] 23. 1.

[b] 32. 1.

[c] 4. 6.

[d] 11. 5.

[e] 9. 5.

[f] 4. 1.

[g] Hyp.

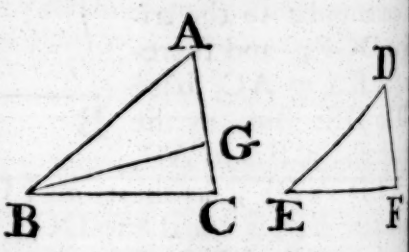
PROP.

PROP. VII. THEOR.

See N. IF two triangles have one angle of the one equal to one angle of the other, and the ſides about two other angles proportionals; then, if each of the remaining angles be either leſs, or not leſs, than a right angle; or if one of them be a right angle: The triangles ſhall be equiangular, and have thoſe angles equal about which the ſides are proportionals.

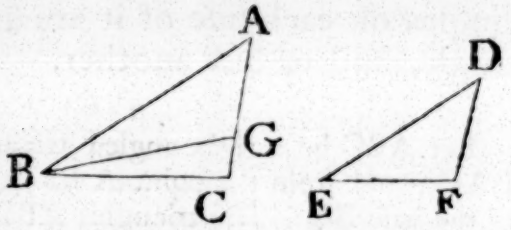
Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the ſides about two other angles ABC, DEF proportionals, ſo that AB is to BC, as DE to EF; and, in the firſt caſe, let each of the remaining angles at C, F be leſs than a right angle. The triangle ABC is equiangular to the triangle DEF, viz. the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

For if the angles ABC, DEF be not equal, one of them is greater than the other; let ABC be the greater, and at the point B in the ſtraight line AB make the angle ABG equal to the angle [a] DEF: And becauſe the angle at A is equal to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal [b] to the remaining angle DFE: Therefore the triangle ABG is equiangular to the triangular DEF; wherefore [c] as AB is to BG, ſo DE to EF; but as DE to EF, ſo, by hypotheſis, is AB to BC; therefore as AB to BC, ſo is AB to BG [d]; and becauſe AB has the ſame ratio to each of the lines BC, BG; BC is equal [e] BG, and therefore the angle BGC is equal to the angle BCG [f]. But the angle BCG is, by hypotheſis, leſs than a right angle; therefore alſo the angle BGC is leſs than a right angle, and the adjacent angle AGB muſt be greater than a right angle [g]. But it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle: But, by hypotheſis, it is leſs than a right angle; which is abſurd. The



[a] 23. 1.
[b] 32. 1.
[c] 4. 6.
[d] 11. 5.
[e] 9. 5.
[f] 5. 1.
[g] 13. 1.

e the angles ABC, DEF are not unequal, that is, they are ual: And the angle at A is equal to the angle at D; wherefore remaining angle at C is equal to the remaining angle at F: erefore the triangle ABC is equiangular to the triangle DEF. Next, Let each of the angles at C, F be not leſs than a right gle: The triangle ABC is alſo in this caſe equiangular to the ngle DEF.

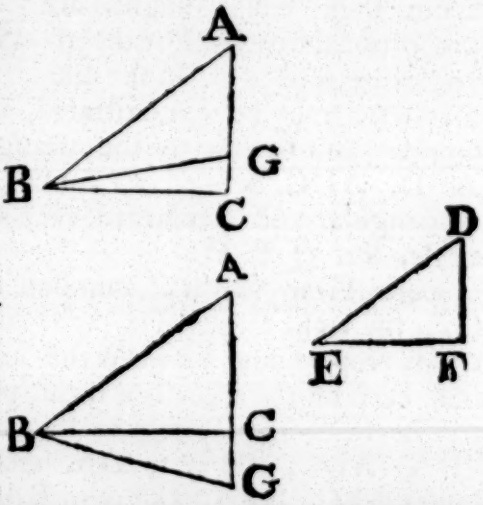


The ſame conſtruction ng made, it may be pro- in like manner that BC equal to BG, and the gle at C equal to the an- BGC: But the angle C is not leſs than a right gle; therefore the angle GC is not leſs than a right angle: Wherefore two angles of triangle BGC are together not leſs than two right angles. ich is impoſſible [h]; and therefore the triangle ABC may be oved to be equiangular to the triangle DEF, as in the firſt e.

[h] 17. 1.

Laſtly, Let one of the angles at C, F, viz. the angle at C, a right angle; in this caſe likewiſe the triangle ABC is e- iangular to the triangular DEF.

For, if they be not equian- lar, make at the point B of ſtraight line AB the angle BG equal to the angle DEF; n it may be proved, as in firſt caſe, that BG is e- al to BC; But the angle CG is a right angle, there- e [f] the angle BGC is alſo a ht angle; whence two of angles of the triangle BGC together not leſs than two ht angles, which is impoſ- le [h]: Therefore the triangle BC is equiangular to the ngle DEF. Wherefore, if two triangles, &c. Q. E. D.



[f] 5. 1.

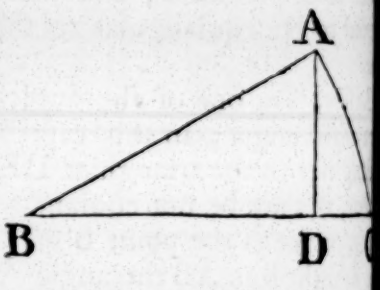
PROP.

PROP. VIII. THEOR.

See N. IN a right angled triangle, if a perpendicular be drawn from the right angle to the baſe; the triangles on each ſide of it are ſimilar to the whole triangle, and to one another.

Let ABC be a right angled triangle having the right angle BAC; and from the point A let AD be drawn perpendicular to the baſe BC: The triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another.

Becauſe the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD [a]: Therefore the triangle ABC is equiangular to the triangle ABD, and the ſides about their equal angles are proportionals [b]; wherefore the triangles are ſimilar [c]: In the like manner it may be demonſtrated, that the triangle ADC is equiangular and ſimilar to the triangle ABC: And the triangles ABD, ADC, being both equiangular and ſimilar to ABC, are equiangular and ſimilar to each other. Therefore in a right angle, &c. Q. E. D.



[a] 32. 1.
[b] 4. 6
[c] 1. def. 6.

Cor. From this it is manifeſt that the perpendicular drawn from the right angle of a right angled triangle to the baſe, is a mean proportional between the ſegments of the baſe: And alſo that each of the ſides is a mean proportional between the baſe and its ſegment adjacent to that ſide: Becauſe, in the triangles BDA, ADC, BD is to DA, as DA to DC [b]; and in the triangles ABC, DBA, BC is to BA, as BA to BD [b]; and in the triangles ABC, ACD, BC is to CA, as CA to CD [b].

PRO

PROP. IX. PROB.

FROM a given ſtraight line to cut off any part required. See N.

Let AB be the given ſtraight line; it is required to cut off part from it.

From the point A draw a ſtraight line AC making any angle with AB; and in AC take any point D, and take AC the ſame multiple of AD that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it: Then AE is the part required to be cut off.

Becauſe ED is parallel to one of the ſides of the triangle ABC, viz. to BC, as CD is to DA, ſo [a] BE to EA; and, by compoſition [b], CA is to AD, as BA to AE: But CA is a multiple of AD; therefore [c] BA is the ſame multiple of AE: Whatever part therefore CD is of AC, AE is the ſame part of AB: Wherefore, from the ſtraight line AB the part required is cut off. Which was to be done.

[a] 2. 6.
[b] 18. 5.
[c] D. 5.

PROP. X. PROB.

TO divide a given ſtraight line ſimilarly to a given divided ſtraight line, that is, into parts that ſhall have the ſame ratios to one another which the parts of the divided given ſtraight line have.

Let AB be the ſtraight line given to be divided, and AC the divided line; it is required to divide AB ſimilarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed ſo as to contain any angle, and join BC, and through the points D, E draw [a] DF, EG parallels to it; and through D draw DHK parallel to AB: Therefore each of the figures FH, HB is a parallellogram; wherefore DH is equal [b] to FG, and HK

[a] 31. 1.
[b] 34. 1.

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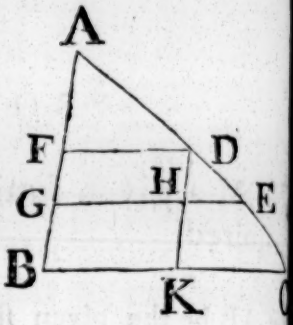
HK to GB: And becauſe HE is parallel to KC one of the ſides of the triangle DKC, as CE to ED, ſo is [c] KH to HD: But KH is equal to BG, and HD to GF: therefore as CE to ED, ſo is BG to GF: Again, becauſe FD is parallel to EG one of the ſides of the triangle AGE, as ED to DA, ſo is GF to FA: But it has been proved that that CE is to ED, as BG to GF; and as ED to DA, ſo GF to FA: Thereſ the given ſtraight line AB is divided ſimilarly to AC. Wh was to be done.

[c] 2. 6.



PROP. XI. PROB.

TO find a third proportional to two giv ſtraight lines.

Let AB, AC be the two given ſtraight lines, and let th be placed ſo as to contain any angle; it is required to find a third proportional to AB, AC.

Produce AB, AC to the points D, E; and make BD equal to AC; and having joined BC, through D, draw DE parallel to it [a].

[a] 31. 1.

Becauſe BC is parallel to DE, a ſide of the triangle ADE, AB is [b] to BD, as AC to CE: But BD is equal to AC; as therefore AB to AC, ſo is AC to CE. Wherefore to the two g ſtraight lines AB, AC a third proportional CE is found. Wh was to be done.

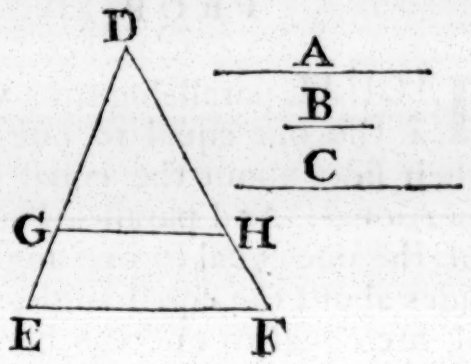
[b] 2. 6.

PROP. XII. PROB.

TO find a fourth proportional to three giv ſtraight lines.

Let A, B, C be the three given ſtraight lines; it is re red to find a fourth proportional to A, B, C.

Take two ſtraight lines DE, DF containing any angle EDF; and upon theſe make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel [a] to it through the point E: And becauſe GH is parallel to EF, one of the ſides of the triangle DEF, DG is to GE, as DH to HF [b]; but DG is equal to A, GE to B, and DH to C; Therefore as A is to B, ſo is C to HF. Wherefore to the three given ſtraight lines A, B, C a fourth proportional HF is found. Which was to be done.



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[a] 31. 1.

[b] 2. 6.

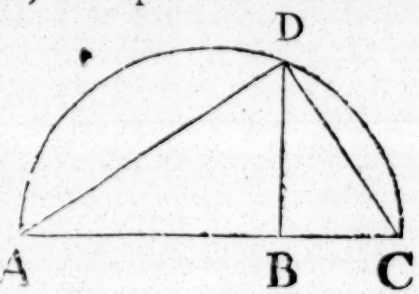
PROP. XIII. PROB.

TO find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines; it is required to find a mean proportional between them.

Place AB, BC in a ſtraight line, and upon AC deſcribe the ſemicircle ADC, and from the point B draw [a] BD at right angles to AC, and join AD, DC.

Becauſe the angle ADC in a ſemicircle is a right angle [b], and becauſe in the right angled triangle ADC, DB is drawn from the right angle perpendicular to the baſe, DB is a mean proportional between AB, BC the ſegments of the baſe [c]: Therefore between the two given ſtraight lines AB, BC, a mean proportional DB is found. Which was to be done.



[a] 11. 1.

[b] 31. 3.

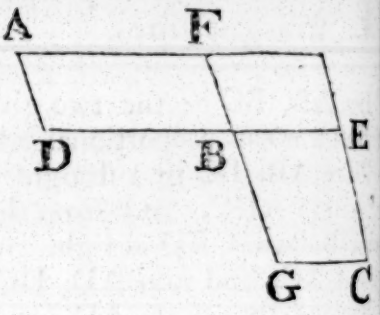
[c] Cor. 8. 6.

PROP. XIV. THEOR.

EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: And parallelograms that have one angle of the one equal to one angle of the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms which have the angles at B equal, and let the ſides DB, BE be placed in the ſame
[a] 14. 1. ſtraight line; wherefore alſo FB, BG are in one ſtraight line[a]: The ſides of the parallelograms AB, BC about the equal angles are reciprocally proportional; that is, DB is to BE, as GB to BF.

Complete the parallelogram FE; and becauſe the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to
[b] 7. 5. FE[b]: But as AB to FE, ſo is
[c] 1. 6. the baſe DB to BE[c]; and as BC to FE, ſo is the baſe GB to BF; therefore as DB to BE,
[d] 11. 5. ſo is GB to BF[d]. Wherefore the ſides of the parallelograms AB, BC about their equal angles are reciprocally proportional.



But let the ſides about the equal angles be reciprocally proportional, viz. as DB to BE, ſo GB to BF; the parallelogram AB is equal to the parallelogram BC.

Becauſe as DB is to BE, ſo GB to BF; and as DB to BE, ſo is the parallelogram AB to the parallelogram FE; and as GB to BF, ſo is the parallelogram BC to the parallelogram FE; therefore as AB to FE, ſo BC to FE[d]: Wherefore the parallelo-
[d] 9. 5. gram AB is equal[e] to the parallelogram BC. Therefore equal parallelograms, &c. Q. E. D.

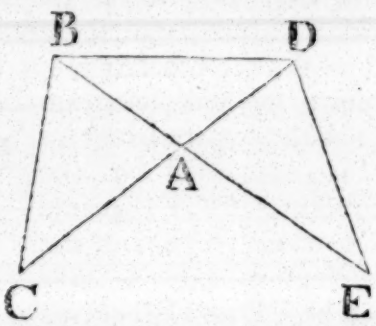
PROP.

PROP. XV. THEOR.

QUAL triangles which have one angle of the one equal to one angle of the other, have their ſides t the equal angles reciprocally proportional: triangles which have one angle in the one equal ne angle in the other, and their ſides about the l angles reciprocally proportional, are equal to another.

t ABC, ADE be equal triangles which have the angle equal to the angle DAE; the ſides about the equal angles triangles are reciprocally proportional; that is, CA is to as EA to AB.

t the triangles be placed ſo that their ſides CA, AD be in ſtraight line; wherefore alſo EA and AB are in one nt line [a]; and join BD. Becauſe the triangle ABC is e- [a] 14. 1.



o the triangle ADE, and that is another triangle; there- s the triangle CAB is to the le BAD, ſo is triangle EAD ngle DAB [b]: But as triangle to triangle BAD, ſo is the CA to AD [c]; and as triangle to triangle DAB, ſo is the EA to AB [c]; as therefore o AD, ſo is EA to AB [d]: [b] 7. 5. [c] 1. 6. [d] 11. 5.

fore the ſides of the triangles ABC, ADE about the equal s are reciprocally proportional.

t let the ſides of the triangles ABC, ADE about the equal s be reciprocally proportional, viz. CA to AD, as EA to the triangle ABC is equal to the triangle ADE.

ving joined BD as before; becauſe as CA to AD, ſo is EA to and as CA to AD, ſo is triangle BAC to triangle BAD [c]; s EA to AB ſo triangle EAD to triangle BAD [c]; there- as triangle BAC to triangle BAD, ſo is triangle EAD to le BAD; that is, the triangles BAC, EAD have the ſame o the triangle BAD: Wherefore the triangle ABC is e- to the triangle ADE. Therefore equal triangles, &c. [e] 9. 5.

D.

PROP. XVI. THEOR.

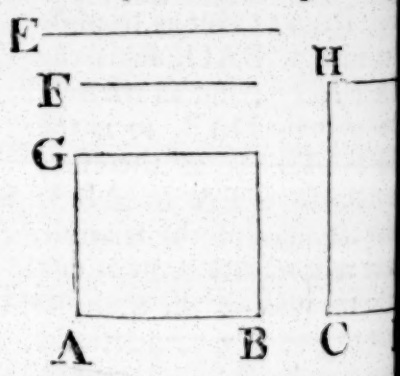
IF four ſtraight lines be proportionals, the rectang contained by the extremes is equal to the recta gle contained by the means: And if the rectang contained by the extremes be equal to the rectang contained by the means, the four ſtraight lines a proportionals.

Let the four ſtraight lines AB, CD, E, F be proportion viz. as AB to CD, ſo E to F; the rectangle contained by A F is equal to the rectangle contained by CD, E.

[a] 11. 1. From the points A, C draw [a] AG, CH at right angles AB, CD; and make AG equal to F, and CH equal to E, a complete the parallelograms BG, DH: Becauſe as AB to C ſo is E to F; and that E is equal to CH, and F to AG; a

[b] 7. 5. is [b] to CD, as CH to AG: Therefore the ſides of the paralle grams BG, DH about the equal angles are reciprocally p portional; but parallelograms which have their ſides abou

[c] 14. 6. qual angles reciprocally proportional, are equal to one another therefore the parallelogram BG is equal to the parallelog DH: And the parallelogram BG is contained by the ſtraight lines AB, F; becauſe AG is equal to F; and the parallelogram DH is contained by CD and E; becauſe CH is equal to E: Therefore the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD and E.



And if the rectangle contained by the ſtraight lines AB be equal to that which is contained by CD, E; theſe four a are proportionals, viz. AB is to CD, as E to F.

The ſame conſtruction being made, becauſe the recta contained by the ſtraight lines AB, F is equal to that w is contained by CD, E, and that the rectangle BG is conta by AB, F, becauſe AG is equal to F; and the rectangle by CD, E, becauſe CH is equal to E; therefore the paral gram BG is equal to the parallelogram DH; and they a

quiang

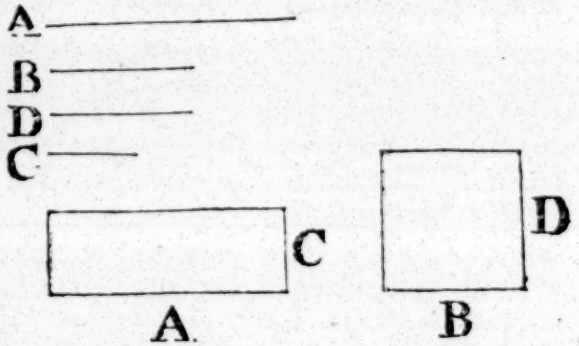
angular: But the ſides about the equal angles of equal pa- Book VI.
lograms are reciprocally proportional [c]: Wherefore as AB
D, ſo is CH to AG; and CH is equal to E, and AG to F: [c] 14. 6.
herefore AB is to CD, ſo E to F. Wherefore, if four, &c.
E. D.

PROP. XVII. THEOR.

three ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the ſquare the mean: And if the rectangle contained by the tremes be equal to the ſquare of the mean, the three aight lines are proportionals.

Let the three ſtraight lines A, B, C be proportionals, viz. A to B, ſo B to C; the rectangle contained by A, C is equal the ſquare of B.

Take D equal to B; and becauſe as A to B, ſo B to C, and t B is equal to D; A is [a] to B, as D to C: But if four ſtraight [a] 7. 5.
s be proportionals, rectangle contain- by the extremes is ual to that which is ntained by the eans [b]: Therefore the ftangle contained by , C is equal to that ntained by B, D: t the rectangle con- ned by B, D is the



[b] 16. 6.

uare of B; becauſe B is equal to D: Therefore the rectangle ntained by A, C is equal to the ſquare of B.

And if the rectangle contained by A, C be equal to the uare of B; A is to B, as B to C.

The ſame conſtruction being made, becauſe the rectangle ntained by A, C is equal to the ſquare of B, and the ſquare B is equal to the rectangle contained by B, D, becauſe B is ual to D; therefore the rectangle contained by A, C is equal that contained by B, D: But if the rectangle contained by e extremes be equal to that contained by the means, the four aight lines are proportionals [b]: Therefore A is to B, as D to

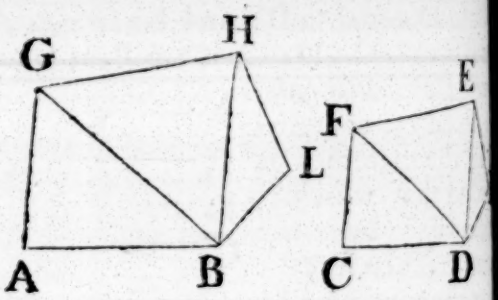
Book VI. C; but B is equal to D; wherefore as A to B, ſo B t
Therefore, if three ſtraight lines, &c. Q. E. D.

PROP. XVIII. PROB.

UPON a given ſtraight line to deſcribe a rectili figure ſimilar, and ſimilarly ſituated to a gi rectilineal figure.

Let AB be the given ſtraight line, and CDEF the given tilineal figure of four ſides; it is required upon the g ſtraight line AB to deſcribe a rectilineal figure ſimilar, an milarly ſituated to CDEF.

Join DF, and at the points A, B in the ſtraight line
[a] 23. 1. make [a] the angle BAG equal to the angle at C, and the a ABG equal to the angle CDF; therefore the remaining a
[b] 32. 1. CFD is equal to the remaining angle AGB [b]: Wherefore triangle FCD is equiangular to the triangle GAB: Again, at the points G, B in the ſtraight line GB make [a] the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle G and the triangle FDE equiangular to the triangle GBH: T becauſe the angle AGB is equal to the angle CFD, and B to DFE, the whole angle AGH is equal to the whole C For the ſame reaſon, the angle ABH is equal to the angle CD alſo the angle at A is equal to the angle at C, and the a GHB to FED: Therefore the rectilineal figure ABHG is e angular to CDEF: But likewiſe theſe figures have their ſide bout the equal angles proportionals: Becauſe the triangles G



[c] 4. 6. FCD being equiangular, BA is [c] to AG, as DC to CF; becauſe AG is to GB, as CF to FD; and as GB to GH, by reaſon of the equiangular triangles BGH, DFE, is FD
[d] 22. 5. FE; therefore, ex aequali [d], AG is to GH, as CF to FE: the ſame manner it may be proved that AB is to BH, as CD DE: And GH is to HB, as FE to ED [c]. Wherefore, beca

e rectilineal figures ABHG, CDEF are equiangular, and ve their ſides about the equal angles proportionals, they are milar to one another [c]. [c] 1. def. 6.

Next, Let it be required to deſcribe upon a given ſtraight line B, a rectilineal figure ſimilar, and ſimilarly ſituated to the ctilineal figure CDKEF of five ſides.

Join DE, and upon the given ſtraight line AB deſcribe the ctilineal figure ABHG ſimilar, and ſimilarly ſituated to the uadrilateral figure CDEF, by the former caſe; and at the oints B, H in the ſtraight line BH, make the angle HBL e- ual to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the re- maining angle at L: And becauſe the figures ABHG, CDEF re ſimilar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is e- ual to the whole angle FEK: For the ſame reaſon, the angle ABL is equal to the angle CDK: Therefore the five ſided fi- gures AGHLB, CFEKD are equiangular: And becauſe the fi- gures AGHB, CFED are ſimilar, GH is to HB, as FE to ED; nd as HB to HL, ſo is ED to EK [c]; therefore, ex æquali [d], [c] 4. 6. GH is to HL, as FE to EK: For the ſame reaſon, AB is to BL, [d] 22. 5. as CD to DK: And BL is to LH, as [c] DK to KE, becauſe the triangles BLH, DKE are equiangular: Therefore, becauſe the five ſided figures AGHLB, CFEKD are equiangular, and have their ſides about the equal angles proportionals, they are ſimilar to one another: And in the ſame manner a rectilineal figure of ſix or more ſides may be deſcribed upon a given ſtraight line ſi- milar to one given, and ſo on. Which was to be done.

PROP. XIX. THEOR.

SIMILAR triangles are to one another in the dupli- cate ratio of their homologous ſides.

Let ABC, DEF be ſimilar triangles having the angle B equal to the angle E, and let AB be to BC, as DE to EF, ſo that the ſide BC is homologous to EF [a]: The triangle ABC has to the [a] 12. def. triangle DEF, the duplicate ratio of that which BC has to EF. 5.

Take BG a third proportional to BC, EF [b], ſo that BC is to [b] 11. 6. EF, as EF to BG, and join GA: Then, becauſe as AB to BC, ſo DE to EF; alternately [c], AB is to DE, as BC to EF: But [c] 16. 5.

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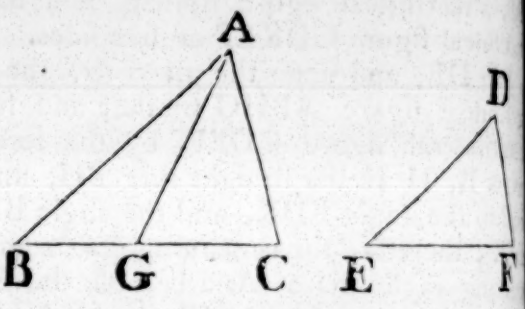
as BC to EF, ſo is EF to BG; therefore [d] as AB to DE, ſo EF to BG: Wherefore the ſides of the triangles ABG, DE which are about the equal angles are reciprocally proportional. But triangles which have the ſides about two equal angles reciprocally proportional are equal to one another [e]: Therefore the triangle ABG is equal to the triangle DEF: And becauſe as BC is to EF, ſo EF to BG; and that if three ſtraight lines be proportionals, the firſt is ſaid [f] to have to the third the duplicate ratio of that which it has to the ſecond; BC therefore has to BG the duplicate ratio of that which BC has to EF: But as BC to BG, ſo is [g] the triangle ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG, the duplicate ratio of that which BC has to EF. But the triangle ABG is equal to the triangle DEF; wherefore alſo the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore ſimilar triangles, &c. Q. E. D.



[d] 11. 5.

[e] 15. 6.

[f] 10. def. 5.

[g] 1. 6.

Cor. From this it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any triangle upon the firſt to a ſimilar, and ſimilarly deſcribed triangle upon the ſecond.

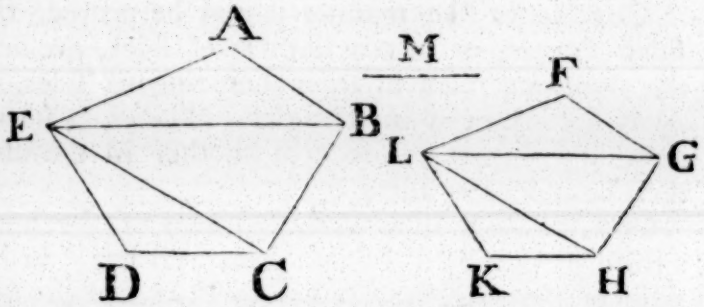
PROP. XX. THEOR.

SIMILAR polygons may be divided into the ſame number of ſimilar triangles, having the ſame ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous ſides have.

Let ABCDE, FGHKL be ſimilar polygons, and let AB be the homologous ſide to FG: The polygons ABCDE, FGHKL may be divided into the ſame number of ſimilar triangles whereof each to each has the ſame ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the ſide FG.

Join BE, EC, GL, LH: And becauſe the polygon ABCDE is ſimilar

ilar to the polygon FGHKL, the angle BAE is equal to the Book VI.
le GFL [a], and BA is to AE, as GF to FL [a]: Wherefore,
auſe the triangles ABE, FGL have an angle in one equal [a] 1. Def.
n angle in the other, and their ſides about theſe equal angles 6.
portionals, the triangle ABE is equiangular [b], and there- [b] 6. 6.
ſimilar to the triangle FGL [c]; wherefore the angle ABE [c] 4. 6.
qual to the angle FGL: And, becauſe the polygons are ſi-
lar, the whole angle ABC is equal [a] to the whole angle FGH;
refore the remaining angle EBC is equal to the remaining
le LGH: And becauſe the triangles ABE, FGL are ſimilar,
is to BA, as LG to GF [a]; and alſo, becauſe the polygons
ſimilar, AB is to BC, as FG to GH [a]; therefore, ex ae-
li [d], EB is to BC, as LG to GH; that is, the ſides about [d] 22. 5.
equal angles EBC, LGH are proportionals; therefore [b]
triangle EBC is equiangular to the triangle LGH, and ſi-
ar to it:



For the
e reaſon,
triangle
D like-
e is ſimi-
to the tri-
le LHK:
erefore the
ilar poly-
ns ABCDE, FGHKL are divided into the ſame number of
ilar triangles.

Alſo theſe triangles have, each to each, the ſame ratio which
polygons have to one another, the antecedents being
BE, EBC, ECD, and the conſequents FGL, LGH, LHK:
d the polygon ABCDE has to the polygon FGHKL the du-
cate ratio of that which the ſide AB has to the homologous
e FG.

Becauſe the triangle ABE is ſimilar to the triangle FGL,
BE has to FGL the duplicate ratio [e] of that which the ſide [e] 19. 6.
has to the ſide GL: For the ſame reaſon, the triangle BEC
s to GLH the duplicate ratio of that which BE has to GL:
erefore, as the triangle ABE to the triangle FGL, ſo [f] is the [f] 11. 5.
angle BEC to the triangle GLH. Again, becauſe the tri-
le EBC is ſimilar to the triangle LGH, EBC has to LGH
duplicate ratio of that which the ſide EC has to the ſide
H: For the ſame reaſon, the triangle ECD has to the triangle
LHK,

Book VI. LHK, the duplicate ratio of that which EC has to LH: therefore the triangle EBC to the triangle LGH, ſo is
[f] 11. 5. triangle ECD to the triangle LHK: But it has been proved the triangle EBC is likewiſe to the triangle LGH, as the trian ABE to the triangle FGL. Therefore, as the triangle ABE is the triangle FGL, ſo is triangle EBC to triangle LGH, and tri gle ECD to triangle LHK: And therefore as one of the an cedents to one of the conſequents, ſo are all the antecedents
[g] 12. 5. all the conſequents [g]. Wherefore as the triangle ABE to triangle FGL, ſo is the polygon ABCDE to the poly FGHKL: But the triangle ABE has to the triangle FGL, duplicate ratio of that which the ſide AB has to the homolog ſide FG. Therefore alſo the polygon ABCDE has to the po gon FGHKL the duplicate ratio of that which AB has to the mologous ſide FG. Wherefore ſimilar polygons, &c. Q. E. D.

Cor. 1. In like manner it may be proved, that ſimilar ſided figures, or of any number of ſides, are one to another the duplicate ratio of their homologous ſides, and it has ready been proved in triangles. Therefore univerſally, ſim rectilineal figures are to one another in the duplicate ratio their homologous ſides.

Cor. 2. And if to AB, FG, two of the homologous ſi
[h] 10 Def. 5. a third proportional M be taken, AB has [h] to M the dupli ratio of that which AB has to FG: But the four ſided fig or polygon upon AB has to the four-ſided figure or polygon on FG likewiſe the duplicate ratio of that which AB has to M. Therefore as AB is to M, ſo is the figure upon AB to the fig
[i] Cor. 19. 6. upon FG, which was alſo proved in triangles [i]. Therefore univerſally, it is manifeſt, that if three ſtraight lines be pro tionals, as the firſt is to the third, ſo is any rectilineal figure u the firſt, to a ſimilar and ſimilarly deſcribed rectilineal figure on the ſecond.

PR

PROP. XXI. THEOR.

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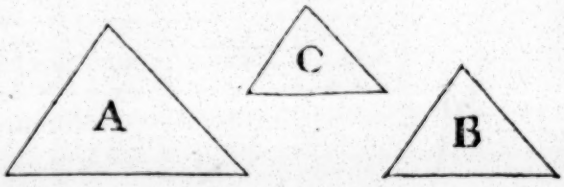
RECTILINEAL figures which are ſimilar to the ſame rectilineal figure, are alſo ſimilar to one another.

Let each of the rectilineal figures A, B be ſimilar to the rectilineal figure C: The figure A is ſimilar to the figure B.

Becauſe A is ſimilar to C, they are equiangular, and alſo have their ſides about the equal angles proportionals [a]. Again, becauſe B is ſimilar to C, they are equiangular, and have their ſides about the equal angles proportionals [a]: Therefore the figures A, B are each of them equiangular to C, and have the ſides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular [b], and have their ſides about the equal angles proportionals [c]. Therefore A is ſimilar [a] to B. Q. E. D.

[a] 1. Def. 6.



[b] 1. Ax. 1.
[c] 11. 5.

PROP. XXII. THEOR.

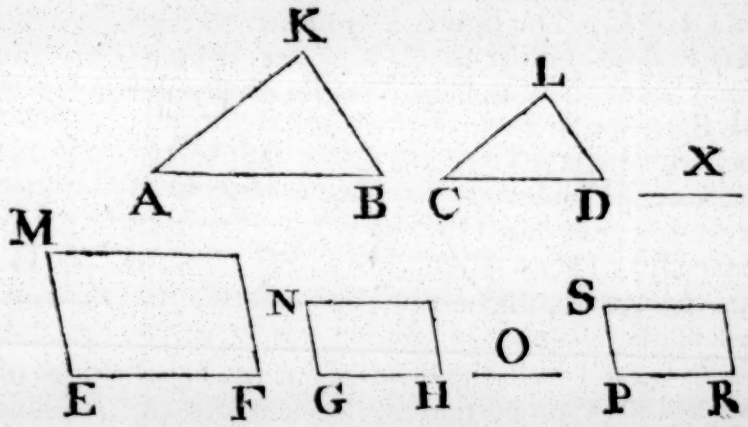
IF four ſtraight lines be proportionals, the ſimilar rectilineal figures ſimilarly deſcribed upon them ſhall alſo be proportionals: And if the ſimilar rectilineal figures ſimilarly deſcribed upon four ſtraight lines be proportionals, thoſe ſtraight lines ſhall be proportionals.

Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz AB to CD, as EF to GH, and upon AB, CD let the ſimilar rectilineal figures KAB, LCD be ſimilarly deſcribed; and upon EF, GH the ſimilar rectilineal figures MF, NH, in like manner: The rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD take a third proportional [a] X; and to EF, GH a third proportional O: And becauſe AB is to CD, as EF to GH, therefore CD is [b] to X, as GH to O; wherefore, ex aequali [c], as AB to X, ſo EF to O: But as AB to X, ſo is [d] the rectilineal

[a] 11. 6.
[b] 11. 5.
[c] 22. 5.
[d] 2 Cor. 20. 6.

Book VI. rectilineal KAB to the rectilineal LCD, and as EF to O, so
[d] the rectilineal MF to the rectilineal NH: Therefore as KA
[d] 2. Cor. to LCD, so [b] is MF to NH.
20. 6. And if the rectilineal KAB be to LCD, as MF to NH; th
[b] 11. 5. ſtraight line AB is to CD, as EF to GH.
[c] 12. 6. Make [c] as AB to CD, ſo EF to PR, and upon PR deſcribe
[f] 18. 6. the rectilineal figure SR ſimilar and ſimilarly ſituated to eith



of the figures MF, NH: Then, becauſe as AB to CD, ſo is E
to PR, and that upon AB, CD are deſcribed the ſimilar an
ſimilarly ſituated rectilineals KAB, LCD, and upon EF, PR
in like manner, the ſimilar rectilineals MF, SR; KAB is t
LCD, as MF to SR; but, by the hypotheſis, KAB is to LCD
as MF to NH; and therefore the rectilineal MF having th
[g] 9. 5. ſame ratio to each of the two NH, SR, theſe are equal [g] to on
another: They are alſo ſimilar, and ſimilarly ſituated; there
fore GH is equal to PR: And becauſe as AB to CD, ſo is E
to PR, and that PR is equal to GH; AB is to CD, as EF t
GH. If therefore four ſtraight lines, &c. Q. E. D.

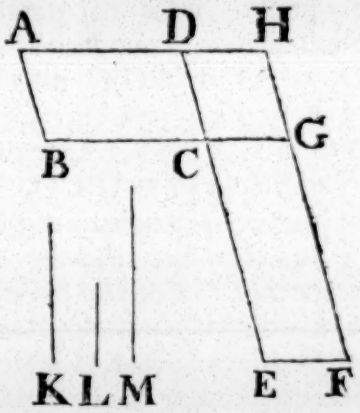
PROP. XXIII. THEOR.

See N. EQUIANGULAR parallelograms have to one ano
ther the ratio which is compounded of the ra
tios of their ſides.

Let AC, CF be equiangular parallelograms, having the ang
BCD equal to the angle ECG: The ratio of the parallelogra
AC to the parallelogram CF, is the ſame with the ratio which
compounded of the ratios of their ſides.

L

Let BC, CG be placed in a ſtraight line; therefore DC and CE are alſo in a ſtraight line [a]; and complete the parallelogram DG; and, taking any ſtraight line K, make [b] as BC to CG, ſo K to L; and as DC to CE, ſo make [b] L to M: Therefore the ratios of K to L, and L to M are the ſame with the ratios of the ſides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is ſaid to be compounded [c] of the ratios of K to L, and L to M: Wherefore alſo K has to M, the ratio compounded of the ratios of the ſides: And becauſe as BC to CG, ſo is the parallelogram AC to the parallelogram CH [d]; but as BC to CG, ſo is K to L; therefore K is [e] to L, as the parallelogram AC to the parallelogram CH: Again, becauſe as DC to CE, ſo is the parallelogram CH to the parallelogram CF; but as DC to CE, ſo is L to M; wherefore L is [e] to M, as the parallelogram CH to the parallelogram CF: Therefore, ſince it has been proved, that as K to L, ſo is the parallelogram AC to the parallelogram CH; and as L to M, ſo the parallelogram CH to the parallelogram CF; ex aequali [f], K is to M, as the parallelogram AC to the parallelogram CF: But K has to M the ratio which is compounded of the ratios of the ſides; therefore alſo the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the ſides. Wherefore equiangular parallelograms, &c. Q. E. D.



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[a] 14. 1.
[b] 12. 6.
[c] A. def. 5.
[d] 1. 6.
[e] 11. 5.
[f] 22. 5.

PROP. XXIV. THEOR.

THE parallelograms about the diameter of any parallelogram, are ſimilar to the whole, and to one another. See N.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: The parallelograms EG, HK are ſimilar both to the whole parallelogram ABCD, and to one another.

Becauſe DC, GF are parallels, the angle ADC is equal [a] to the angle AGF: For the ſame reaſon, becauſe BC, EF are pa-

[a] 29. 1.

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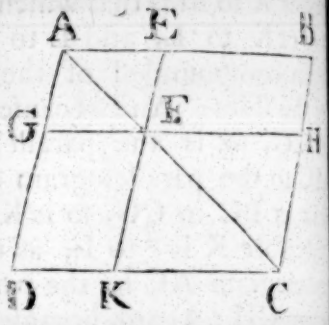
rallels, the angle ABC is equal to the angle AEF: And each of the angles BCD, EFG is equal to the oppoſite angle DAB [b], and therefore are equal to one another; wherefore the parallelograms ABCD, AEFG are equiangular: And becauſe the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to one another; therefore [c] as AB to BC, ſo is AE to EF: And becauſe the oppoſite ſides of parallelograms are equal to one another [b], AB is [d] to AD, as AE to AG; and DC to CB, as GF to FE; and alſo CD to DA, as FG to GA: Therefore the ſides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore ſimilar to one another [e]: For the ſame reaſon, the parallelogram ABCD is ſimilar to the parallelogram FHCK. Wherefore each of the parallelograms GE, KH is ſimilar to DB: But rectilineal figures which are ſimilar to the ſame rectilineal figure, are alſo ſimilar to one another [f]; therefore the parallelogram GE is ſimilar to KH. Wherefore the parallelograms, &c. Q. E. D.

[b] 34. 1.
[c] 4. 6.
[d] 7. 5.
[e] 1. Def. 6.
[f] 21. 6.



PROP. XXV. PROB.

See N.

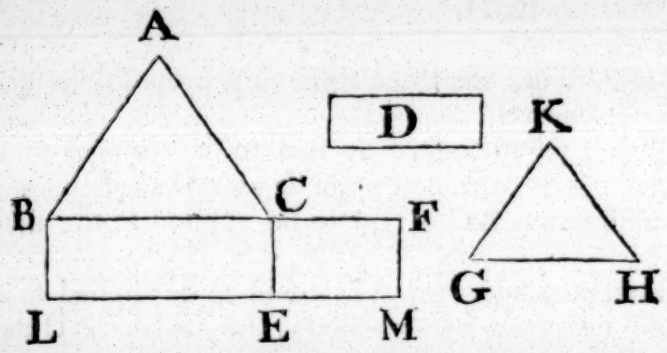
TO deſcribe a rectilineal figure which ſhall be ſimilar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be deſcribed is required to be ſimilar, and D that to which it muſt be equal. It is required to deſcribe a rectilineal figure ſimilar to ABC and equal to D.

Upon the ſtraight line BC deſcribe [a] the parallelogram BE equal to the figure ABC; alſo upon CE deſcribe [a] the parallelogram CM equal to D, and having the angle FCE equal to the angle CBL: Therefore BC and CF are in a ſtraight line [b], as alſo LE and EM: Between BC and CF find [c] a mean proportional GH, and upon GH deſcribe [d] the rectilineal figure KGH ſimilar and ſimilarly ſituated to the figure ABC: And becauſe BC is to GH as GH to CF, and if three ſtraight lines be proportionals, as the firſt is to the third, ſo is [e] the figure

[a] Cor. 45. 1.
[b] 29. 1. 14. 1.
[c] 13. 6.
[d] 18. 6.
[e] 2. Cor. 20. 6.

upon the first to the similar and similarly described fi- Book VI.
upon the second; therefore as BC to CF, so is the rec-
al figure ABC to KGH: But as BC to CF, so is [f] the pa- [f] 1. 6.
ogram BE to the parallelogram EF: Therefore as the rec-
al figure ABC is to KGH, so is the parallelogram BE to
arallelogram EF [g]: And the rectilineal figure ABC is equal [g] 11. 5.

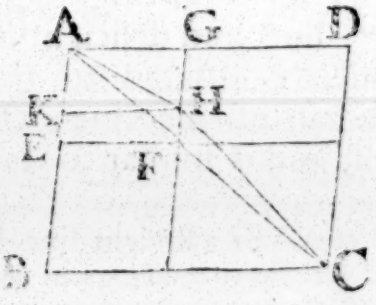


parallelogram BE; therefore the rectilineal figure KGH
ual [h] to the parallelogram EF: But EF is equal to the fi- [h] 14. 5.
D; wherefore also KGH is equal to D; and it is similar
BC. Therefore the rectilineal figure KGH has been de-
d similar to the figure ABC, and equal to D. Which was
done.

PROP. XXVI. THEOR.

two similar parallelograms have a common angle, nd be similarly situated; they are about the e diameter.

t the parallelograms ABCD, AEFG be similar and simi-
situated, and have the angle DAB common. ABCD and
G are about the same diameter.
, if not, let, if possible, the
elogram BD have its dia-
AHC in a different straight
om AF the diameter of the
elogram EG, and let GF
AHC in H; and through
w HK parallel to AD or
Therefore the parallelograms
D, AKHG being about the
diameter, they are similar



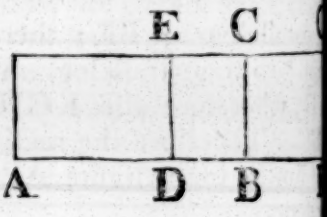
e another [a]: Wherefore as DA to AB, so is [b] GA to AK: [a] 24. 6.
ecause ABCD and AEFG are similar parallelograms, [b] 1. Def.
as 6.

Book VI. as DA is to AB ſo is GA to AE; therefore [c] as GA to AE
GA to AK; wherefore GA has the ſame ratio to each of
[c] 11. 5. ſtraight lines AE, AK; and conſequently AK is equal [d] to
[d] 9. 5. the leſs to the greater, which is impoſſible: Therefore AH
and AKHG are not about the ſame diameter; wherefore AH
and AEFG muſt be about the ſame diameter. Therefore, if
ſimilar, &c. Q. E. D.

'To underſtand the three following propoſitions more e
it is to be obſerved,

'1. That a parallelogram is ſaid to be applied to a ſtra
'line, when it is deſcribed upon it as one of its ſides. Ea
'the parallelogram AC is ſaid to be applied to the ſtraight
'AB.

'2. But a parallelogram AE is ſaid to be applied to a ſtra
'line AB, deficient by a parallelogram, when AD the ba
'AE is leſs than AB, and there-
'fore AE is leſs than the paral-
'lelogram AC deſcribed upon
'AB in the ſame angle, and be-
'tween the ſame parallels, by the
'parallelogram DC; and DC is
'therefore called the defect of
'AE.



'3. And a parallelogram AG is ſaid to be applied to a ſtra
'line AB, exceeding by a parallelogram, when AF the ba
'AG is greater than AB, and therefore AG exceeds AC the
'rallelogram deſcribed upon AB in the ſame angle, and bet
'the ſame parallels, by the parallelogram BG.'

PROP. XXVII. THEOR.

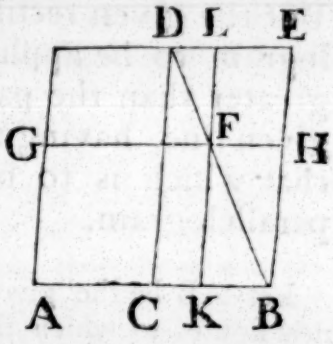
See N. OF all parallelograms applied to the ſame ſtrai
line, and deficient by parallelograms ſimilar
ſimilarly ſituated to that which is deſcribed u
the half of the line; that which is applied to
half, and is ſimilar to its defect, is the greateſt.

Let AB be a ſtraight line divided into two equal parts in
and let the parallelogram AD be applied to the half
which is therefore deficient from the parallelogram upon
whole line AB by the parallelogram CE upon the other
CB: Of all the parallelograms applied to any other part

B, and deficient by parallelograms that are ſimilar, and ſimi- Book VI.
ly ſituated to CE, AD is the greateſt.

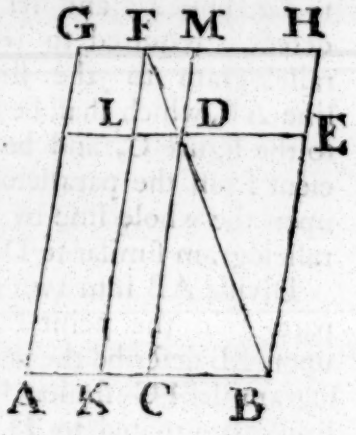
Let AF be any parallelogram applied to AK, any other part AB than the half, ſo as to be deficient from the parallelogram on the whole line AB by the parallelogram KH ſimilar, and ſimilarly ſituated to CE; AD is greater than AF.

Firſt, Let AK the baſe of AF be greater than AC the half AB; and becauſe CE is ſimilar to parallelogram KH, they are about ſame diameter [a]: Draw their diameter DB, and complete the ſcheme: cauſe the parallelogram CF is equal FE, add KH to both, therefore whole CH is equal to the whole : But CH is equal [c] to CG, beſe the baſe AC is equal to the baſe ; therefore CG is equal to KE: each of theſe add CF; then the ole AF is equal to the gnomon CHL: Therefore CE or the allelogram AD is greater than the parallelogram AF.



[a] 26. 6.
[b] 43. 1.
[c] 36. 1.

Next, Let AK the baſe of AF, leſs than AC, and, the ſame ſtruction being made, the paralogram DH is equal to DG [c], for M is equal to MG [d], becauſe BC equal to CA; wherefore DH is ater than LG: But DH is equal [b] DK; therefore DK is greater than : To each of theſe add AL; then whole AD is greater than the ole AF. Therefore of all parallerams applied, &c. Q. E D.

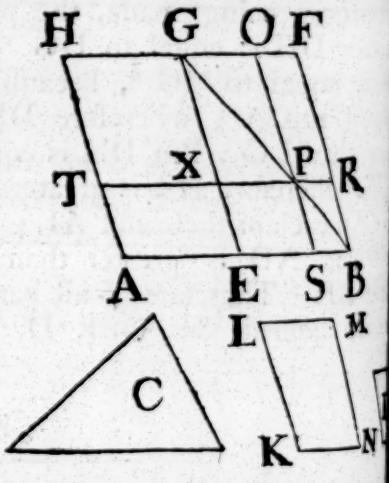


[d] 34. 1.

PROP. XXVIII. PROB.

See N. **TO a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram ſimilar to a given parallelogram. But the given rectilineal figure to which the parallelogram to be applied is to be equal, muſt not be greater than the parallelogram applied to half of the given line, having its defect ſimilar to the defect that which is to be applied; that is, to the given parallelogram.**

Let AB be the given ſtraight line, and C the given rectilineal figure, to which the parallelogram to be applied is required to be equal, which figure muſt not be greater than the parallelogram applied to the half of the line having its defect ſimilar that upon the whole line ſimilar to the defect of that which is to be applied; and let D be the parallelogram to which the defect is required to be ſimilar. It is required to apply a parallelogram to the ſtraight line AB, which ſhall be equal to the figure C, and be deficient from the parallelogram upon the whole line by a parallelogram ſimilar to D.



Divide AB into two equal parts [a] in the point E, and upon EB deſcribe the parallelogram EBFG ſimilar [b] and ſimilarly ſituated to D, and complete the parallelogram AG, which muſt either be equal to C, or greater than it, by the determination: And if AG be equal to C, then what was required is already done. For, upon the ſtraight line AB, the parallelogram AG is applied equal to the figure C, and deficient by the parallelogram ſimilar to D: But if AG be not equal to C, it is greater than it; and EF is equal to AG, therefore EF alſo is greater than C. Make [c] the parallelogram KLMN equal to the excess EF above C, and ſimilar and ſimilarly ſituated to D; but KM is ſimilar to EF, therefore [d] alſo KM is ſimilar to EF: Let

[a] 10. 1.
[b] 18. 6.
[c] 25. 6.
[d] 21. 6.

e homologous ſide to EG, and LM to GF: And becauſe is equal to C and KM together, EF is greater than KM; fore the ſtraight line EG is greater than KL, and GF LM: Make GX equal to LK, and GO equal to LM, and lete the parallelogram XGOP: Therefore XO is equal imilar to KM; but KM is ſimilar to EF; wherefore alſo is ſimilar to EF, and therefore XO and EF are about the diameter [c]: Let GPB be their diameter, and complete the [c] 26. 6.
ne: Then becauſe EF is equal to C and KM together, and a part of the one is equal to KM a part of the other, the inder, viz. the gnomon ERO, is equal to the remainder And becauſe OR is equal [f] to XS, by adding SR to each, [f] 34. 1.
whole OB is equal to the whole XB: But XB is equal [g] to [g] 36. 1.
becauſe the baſe AE is equal to the baſe EB; wherefore TE is equal to OB: Add XS to each, then the whole TS ual to the whole, viz. to the gnomon ERO: But it has been d, that the gnomon ERO is equal to C, and therefore al- S is equal to C. Wherefore the parallelogram TS, equal given rectilineal figure C, is applied to the given ſtraight AB deficient by the parallelogram SR ſimilar to the gi- one D, becauſe SR is ſimilar to EF [h]. Which was to be [h] 24. 6.

PROP. XXIX. PROB.

O a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, exceeding by rallelogram ſimilar to another given. See N.

t AB be the given ſtraight line, and C the given rectili- figure to which the parallelogram to be applied is required e equal, and D the parallelogram to which the exceſs of ne to be applied above that upon the given line is requi- be ſimilar. It is required to apply a parallelogram to the ſtraight line AB which ſhall be equal to the figure C, eding by a parallelogram ſimilar to D.

vide AB into two equal parts in the point E, and upon eſcribe [a] the parallelogram EL ſimilar and ſimilarly ſitu- [a] 18. 6.

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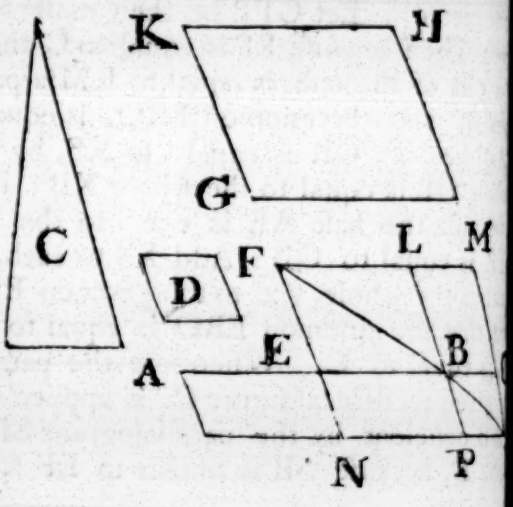
ated to D: and make [b] the parallelogram GH equal to EL C together, and ſimilar and ſimilarly ſituated to D; wh[ere]fore GH is ſimilar to EL [c]: Let KH be the ſide homologous FL, and KG to FE; And becauſe the parallelogram G[H] greater than EL, therefore the ſide KH is greater than [EL] and KG than FE: Produce FL and FE, and make FLM [equal] to KH, and FEN to KG, and complete the parallelogram [MN]. MN is therefore equal and ſimilar to GH; but GH is ſimilar to EL; wherefore MN is ſimilar to EL, and conſequently EL and MN are about the ſame diameter [d]: Draw their diameter FX, and complete the ſcheme. Therefore ſince GH is equal to EL and C together, and that GH is equal to MN; MN is equal to EL and C: Take away the common part [EL], then the remainder, viz. the gnomon NOL, is equal to C. [e] becauſe AE is equal to EB, the parallelogram AN is equal [to] the parallelogram NB, that is, to BM [f]. Add NO to [each]; therefore the whole, viz. the parallelogram AX, is equal [to] the gnomon NOL. But the gnomon NOL is equal to C; there[fore] alſo AX is equal to C. Wherefore to the ſtraight line [AB] there is applied the parallelogram AX equal to the given [rec]tilineal C, exceeding by the parallelogram PO, which is ſi[mi]lar to D, becauſe PO is ſimilar to EL [g]. Which was to be done.



[b] 25. 6.
[c] 21. 6.
[d] 26. 6.
[e] 36. 1.
[f] 43. 1.
[g] 24. 6.

PROP. XXX. PROB.

TO cut a given ſtraight line in extreme and [mean] ratio.

Let AB be the given ſtraight line; it is required to cut [it] in extreme and mean ratio.

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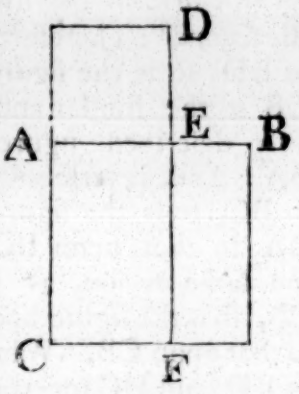
Upon AB deſcribe [a] the ſquare BC, and to AC apply the parallelogram CD equal to BC, exceeding by the figure AD ſimilar to BC [b]: But BC is a ſquare, therefore alſo AD is a ſquare; and becauſe BC is equal to CD, by taking the common part CE from each, the remainder BF is equal to the remainder AD: And theſe figures are equiangular; therefore their ſides about the equal angles are reciprocally proportional [c]: Therefore as FE to ED, ſo AE to EB: But FE is equal to AC [d], that is, to AB; and ED is equal to AE: Therefore as BF to AE, ſo is AE to EB: But AB is greater than AE; wherefore AE is greater than EB [e]: Therefore the ſtraight line AB is cut in extreme and mean ratio in E [f]. Which was to be done.

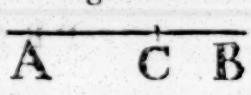


[a] 46. 1.
[b] 29. 6.
[c] 14. 6.
[d] 34. 1.
[e] 14. 5.
[f] 3. Def. 6.

Otherwiſe,

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Divide AB in the point C, ſo that the rectangle contained by AB, BC be equal to the ſquare of AC [g]. Then, becauſe the rectangle AB, BC is equal to the ſquare of AC, as BA to AC, ſo is AC to CB [h]: Therefore AB is cut in extreme and mean ratio in C [f]. Which was to be done.



[g] 11. 2.
[h] 17. 6.

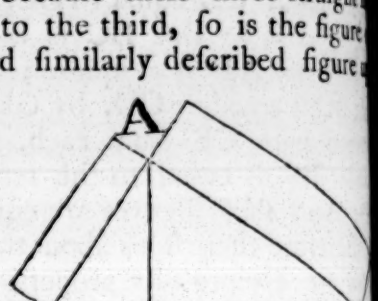
PROP. XXXI. THEOR.

IN right angled triangles, the rectilineal figure deſcribed upon the ſide oppoſite to the right angle, is equal to the ſimilar, and ſimilarly deſcribed figures upon the ſides containing the right angle. See N.

Let ABC be a right angled triangle, having the right angle BAC: The rectilineal figure deſcribed upon BC is equal to the ſimilar, and ſimilarly deſcribed figures upon BA, AC.

Draw the perpendicular AD; therefore, becauſe in the right angled triangle ABC, AD is drawn from the right angle BAC perpendicular to the baſe BC, the triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another [a], [a] 8. 6.

Book VI. and becauſe the triangle ABC is ſimilar to ABD, as CB
BA, ſo is BA to BD [b], and becauſe theſe three ſtraight l
[b] 4. 6. are proportionals, as the firſt to the third, ſo is the figure
on the firſt to the ſimilar, and ſimilarly deſcribed figure u
[c] 2 Cor. 20. 6. the ſecond [c]: Therefore as CB to BD, ſo is the figure upon CB to the ſimilar and ſimilarly deſcribed figure upon BA: And, inverſely [d], as DB to BC, ſo is the figure upon BA to that upon BC: For the ſame reaſon, as DC to CB, ſo is the figure upon CA to that upon CB. Wherefore as BD and DC together to BC, ſo are the figures upon BA,



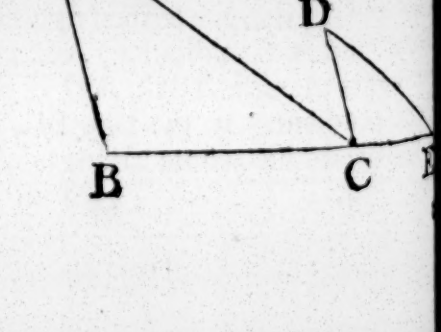
[d] B. 5.
[e] 24. 5. to that upon BC [e]: But BD and DC together are equal to
[f] A. 5. Therefore the figure deſcribed on BC is equal [f] to the ſim
and ſimilarly deſcribed figures on BA, AC. Wherefore
right angled triangles, &c. Q. E. D.

PROP. XXXII. THEOR.

See N. IF two triangles which have two ſides of the
proportional to two ſides of the other, be jo
ed at one angle, ſo as to have their homologo
ſides parallel to one another; the remaining ſi
ſhall be in a ſtraight line.

Let ABC, DCE be two triangles which have the two ſ
BA, AC proportional to the two CD, DE, viz. BA to A
as CD to DE; and let AB be parallel to DC, and AC to
BC and CE are in a ſtraight line.

Becauſe AB is parallel to DC, and the ſtraight line AC meets them, the alternate angles BAC, ACD
[a] 29. 1. are equal [a]; for the ſame reaſon the angle CDE is equal to the angle ACD; wherefore alſo BAC is equal to CDE: And becauſe



triangles ABC, DCE have one angle at A equal to one at and the ſides about theſe angles proportionals, viz. BA to , as CD to DE, the triangle ABC is equiangular [b] to DCE : erefore the angle ABC is equal to the angle DCE : And the le BAC was proved to be equal to ACD : Therefore the ole angle ACE is equal to the two angles ABC, BAC ; add common angle ACB, then the angles ACE, ACB are e- l to the angles ABC, BAC, ACB : But ABC, BAC, ACB equal to two right angles [c] ; therefore alſo the angles ACE, B are equal to two right angles : And ſince at the point in the ſtraight line AC, the two ſtraight lines BC, CE, ich are on the oppoſite ſides of it, make the adjacent angles E, ACB equal to two right angles ; therefore [d] BC and are in a ſtraight line. Wherefore, if two triangles, &c. E. D.

Book VI.

[b] 6. 6.

[c] 32. 1.

[d] 14. 1.

PROP. XXXIII. THEOR.

N equal circles, angles, whether at the centers or circumferences, have the ſame ratio which the rcumferences on which they ſtand have to one an- her : So alſo have the ſectors.

See N.

Let ABC, DEF be equal circles ; and at their centers the gles BGC, EHF, and the angles BAC, EDF at their cir- mferences ; as the circumference BC to the circumference F, ſo is the angle BGC to the angle EHF, and the angle AC to the angle EDF ; and alſo the ſector BGC to the ſector HF.

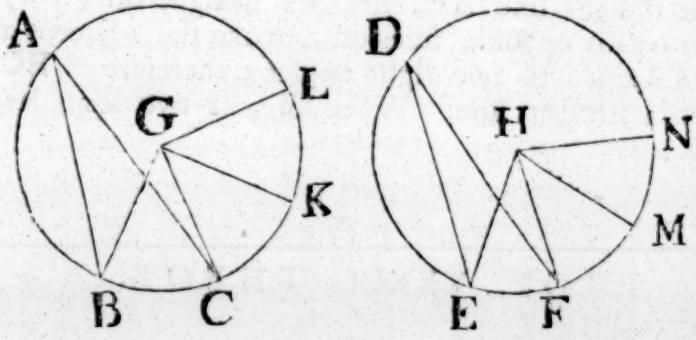
Take any number of circumferences CK, KL, each equal to C, and any number whatever FM, MN each equal to EF : d join GK, GL, HM, HN. Becauſe the circumferences C, CK, KL are all equal, the angles BGC, CGK, KGL e alſo all equal [a] : Therefore what multiple ſoever the circum- rence BL is of the circumference BC, the ſame multiple is e angle BGL of the angle BGC : For the ſame reaſon, what- er multiple the circumference EN is of the circumference F, the ſame multiple is the angle EHN of the angle EHF :

[a] 27. 3.

Book VI.

And if the circumference BL be equal to the circumfere EN, the angle BGL is alſo equal [a] to the angle EHN; a

[a] 27. 3. if the circumference BL be greater than EN, likewiſe the an BGL is greater than EHN; and if leſs, leſs: There being th four magnitudes, the two circumferences BC, EF, and t two angles BGC, EHF; of the circumference BC, and of t angle BGC, have been taken any equimultiples whatever, v the circumference BL, and the angle BGL; and of the circu ference EF, and of the angle EHF, any equimultiples wh



ever, viz. the circumference EN, and the angle EHN: A it has been proved, that if the circumference BL be grea than EN, the angle BGL is greater than EHN; and if qual, equal; and if leſs, leſs: As therefore the circumfere

[b] 5. Def. 5. BC to the circumference EF, ſo [b] is the angle BGC to t angle EHF: But as the angle BGC is to the angle EHF, ſo

[c] 15. [c] the angle BAC to the angle EDF, for each is double

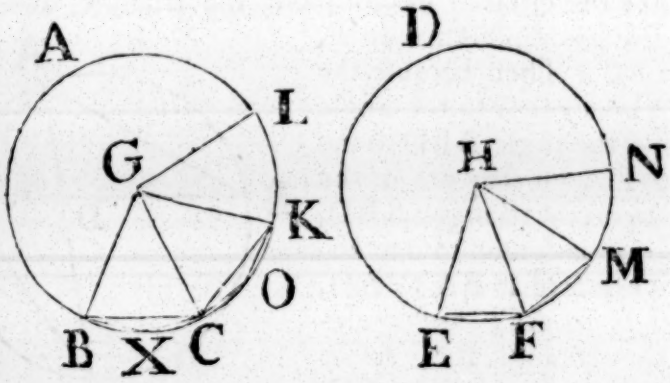
[d] 20. 3. each [d]: Therefore, as the circumference BC is to EF, ſo is t angle BGC to the angle EHF, and the angle BAC to the angl EDF.

Alſo, as the circumference BC to EF, ſo is the ſector BG to the ſector EHF. Join BC, CK, and in the circumferenc BC, CK take any points X, O, and join BX, XC, CO, OK Then, becauſe in the triangles GBC, GCK the two ſides BG GC are equal to the two CG, GK, and that they contain e

[e] 4. 1. qual angles; the baſe BC is equal [e] to the baſe CK, and th triangle GBC to the triangle GCK: And becauſe the circum ference BC is equal to the circumference CK, the remainin part of the whole circumference of the circle ABC, is equal t the remaining part of the whole circumference of the ſam circle: Wherefore the angle BXC is equal to the angle COK

[f] 11. Def. 3. and the ſegment BXC is therefore ſimilar to the ſegment COK [f] an

they are upon equal ſtraight lines BC, CK: But ſimilar ſegments of circles upon equal ſtraight lines, are equal [g] to one another: Therefore the ſegment BXC is equal to the ſegment COK: Book VI. [g] 24. 3.
d the triangle BGC is equal to the triangle CGK; therefore whole, the ſector BGC, is equal to the whole, the ſector CGK: the ſame reaſon the ſector KGL, is equal to each of the tors BGC, CGK: In the ſame manner, the ſectors EHF, M, MHN may be proved equal to one another: Therefore, at multiple ſoever the circumference BL is of the circumfece BC, the ſame multiple is the ſector BGL of the ſector C: For the ſame reaſon, whatever multiple the circumfece EN is of EF, the ſame multiple is the ſector EHN of the tor EHF: And if the circumference BL be equal to EN, the



tor BGL is equal to the ſector EHN; and if the circumfece BL be greater than EN, the ſector BGL is greater than ſector EHN; and if leſs, leſs: Since then, there are four gnitudes, the two circumferences BC, EF, and the two ſec- BGC, EHF, and of the circumference BC, and ſector C, the circumference BL and ſector BGL are any equiltiples whatever; and of the circumference EF and ſector F, the circumference EN and ſector EHN are any equiltiples whatever; and that it has been proved if the circumnce BL be greater than EN, the ſector BGL is greater than ſector EHN; and if equal, equal; and if leſs, leſs. Therefore [b] as the circumference BC is to the circumference EF, ſo the ſector BGC to the ſector EHF. Wherefore, in equal cir-, &c. Q. E. D. [b] 5. def. 5.

PROP.

PROP. B. THEOR.

See N. IF an angle of a triangle be biſected by a ſtraig line, which likewiſe cuts the baſe; the rectan contained by the ſides of the triangle is equal to t rectangle contained by the ſegments of the baſe, t gether with the ſquare of the ſtraight line biſecti the angle.

Let ABC be a triangle, and let the angle BAC be biſect by the ſtraight line AD; the rectangle BA, AC is equal to t rectangle BD, DC, together with the ſquare of AD.

[a] 5. 4. Deſcribe the circle [a] ACB about the triangle, and produ AD to the circumference in E, and join EC: Then becauſe the angle BAD is equal to the angle CAE, and the angle ABD to the

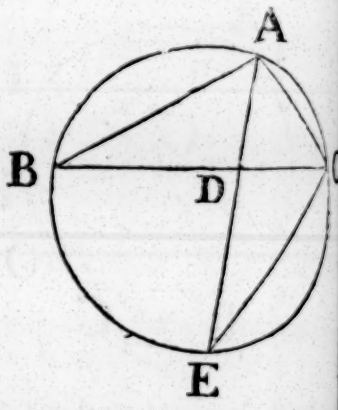
[b] 21. 3. angle [b] AEC, for they are in the ſame ſegment; the triangles ABD, AEC are equiangular to one another: Therefore as BA to AD,

[c] 4. 6. ſo is [c] EA to AC, and conſequently the rectangle BA, AC is

[d] 16. 6. equal [d] to the rectangle EA, AD,

[e] 3. 2. that is [e] to the rectangle ED, DA together with the ſquare of AD: But the rectangle ED, D

[f] 35. 3. is equal to the rectangle [f] BD, DC. Therefore the rectan BA, AC is equal to the rectangle BD, DC, together with t ſquare of AD. Wherefore, if an angle, &c. Q. E. D.



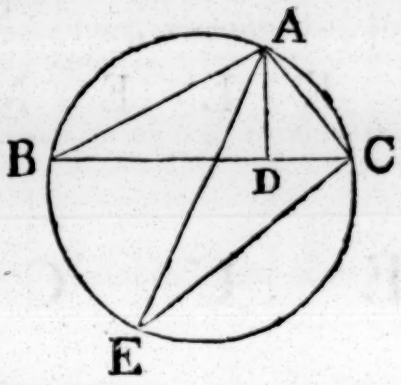
PROP. C. THEOR.

See N. IF from an angle of a triangle a ſtraight line drawn perpendicular to the baſe; the rectan contained by the ſides of the triangle is equal to t rectangle contained by the perpendicular and the ameter of the circle deſcribed about the triangle.

Let ABC be a triangle, and AD the perpendicular from angle A to the baſe BC; the rectangle BA, AC is equal to rectangle contained by AD and the diameter of the circle ſcribed about the triangle.

Deſcri

Deſcribe [a] the circle ACB about
e triangle, and draw its diame-
r AE, and join EC: Becauſe the
ght angle BDA is equal [b] to the
gle ECA in a ſemicircle, and
e angle ABD to the angle AEC
the ſame ſegment [c]; the tri-
gles ABD, AEC are equian-
lar. Therefore as [d] BA to AD,
is EA to AC, and conſequent-
the rectangle BA, AC is equal [e]
the rectangle EA, AD. If,
erefore, from an angle, &c. Q. E. D.

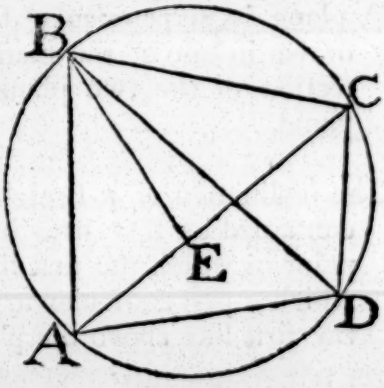


[a] 5. 4.
[b] 31. 3.
[c] 21. 3.
[d] 4 6.
[e] 16. 6.

PROP. D. THEOR.

THE rectangle contained by the diagonals of a quadrilateral inſcribed in a circle, is equal to oth the rectangles contained by its oppoſite ſides.

Let ABCD be any quadrilateral inſcribed in a circle, and in AC, BD; the rectangle contained by AC, BD is equal to e two rectangles contained by AB, CD, and by AD, BC†. Make the angle ABE equal to the angle DBC; add to each theſe the common angle EBD, then the angle ABD is e-ual to the angle EBC: And the angle BDA is equal [a] to the gle BCE, becauſe they are in the ſame ſegment; therefore e triangle ABD is equiangular to e triangle BCE: Wherefore [b] as C is to CE, ſo is BD to DA, and onſequently the rectangle BC, AD equal [c] to the rectangle BD, CE: gain, becauſe the angle ABE is ual to the angle DBC, and the gle [a] BAE to the angle BDC, the iangle ABE is equiangular to the iangle BCD: As therefore BA to E, ſo is BD to DC; wherefore e rectangle BA, DC is equal to the rectangle BD, AE: But e rectangle BC, AD has been ſhewn equal to the rectangle D, CE; therefore the whole rectangle AC, BD is equal to e rectangle AB, DC together with the rectangle AD, BC. herefore the rectangle, &c. Q. E. D.



[a] 21. 3.
[b] 4. 6.
[c] 16. 6.

THE

† This is a Lemma of Cl. Ptolomaeus in page 9. of his μεγαλη συνταξις.

THE ELEMENTS OF EUCLID

BOOK XI.

DEFINITIONS.

I.

A Solid is that which hath length, breadth, and thickne

II.

That which bounds a ſolid is a ſuperficies.

III.

A ſtraight line is perpendicular, or at right angles to a pla when it makes right angles with every ſtraight line meeti it in that plane.

IV.

A plane is perpendicular to a plane, when the ſtraight lin drawn in one of the planes perpendicularly to the comm ſection of the two planes, are perpendicular to the oth plane.

V.

The inclination of a ſtraight line to a plane is the acute ang contained by that ſtraight line, and another drawn from t point in which the firſt line meets the plane, to the point which a perpendicular to the plane drawn from any point the firſt line above the plane, meets the ſame plane.

VI

The inclination of a plane to a plane is the acute angle contai ed by two ſtraight lines drawn from any the ſame point their common ſection at right angles to it, one upon o plane, and the other upon the other plane.

VII. T

VII.

Two planes are ſaid to have the ſame, or a like inclination to one another, which two other planes have, when the ſaid angles of inclination are equal to one another.

VIII.

Parallel planes are ſuch which do not meet one another tho' produced.

IX.

A ſolid angle is that which is made by the meeting of more than two plane angles, which are not in the ſame plane, in one point. See N.

X.

The tenth definition is omitted for reaſons given in the notes.' See N.

XI.

Similar ſolid figures are ſuch as have all their ſolid angles equal, each to each, and which are contained by the ſame number of ſimilar planes. See N.

XII.

A pyramid is a ſolid figure contained by planes that are conſtituted betwixt one plane and one point above it in which they meet.

XIII.

A priſm is a ſolid figure contained by plane figures of which two that are oppoſite, are equal, ſimilar, and parallel to one another; and the others parallelograms.

XIV.

A ſphere is a ſolid figure deſcribed by the revolution of a ſemicircle about its diameter, which remains unmoved.

XV.

The axis of a ſphere is the fixed ſtraight line about which the ſemicircle revolves

XVI.

The center of a ſphere is the ſame with that of the ſemicircle.

XVII.

The diameter of a ſphere is any ſtraight line which paſſes thro' the center, and is terminated both ways by the ſuperficies of the ſphere.

XVIII.

A cone is a ſolid figure deſcribed by the revolution of a right angled triangle about one of the ſides containing the right angle, which ſide remains fixed.

If the fixed ſide be equal to the other ſide containing the right angle,

Book XI.

angle, the cone is called a right angled cone; if it be less than the other ſide, an obtuſe angled, and if greater, an acute angled cone.

XIX.

The axis of a cone is the fixed ſtraight line about which the triangle revolves.

XX.

The baſe of a cone is the circle deſcribed by that ſide containing the right angle, which revolves.

XXI.

A cylinder is a ſolid figure deſcribed by the revolution of a right angled parallelogram about one of its ſides which remains fixed.

XXII.

The axis of a cylinder is the fixed ſtraight line about which the parallelogram revolves.

XXIII.

The baſes of a cylinder are the circles deſcribed by the two revolving oppoſite ſides of the parallelogram.

XXIV.

Similar cones and cylinders are thoſe which have their axes and the diameters of their baſes proportionals.

XXV.

A cube is a ſolid figure contained by ſix equal ſquares.

XXVI.

A tetrahedron is a ſolid figure contained by four equal and equilateral triangles.

XXVII.

An octahedron is a ſolid figure contained by eight equal and equilateral triangles.

XXVIII.

A dodecahedron is a ſolid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An icoſahedron is a ſolid figure contained by twenty equal and equilateral triangles.

DEF. A.

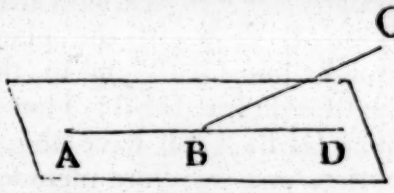
A parallelepiped is a ſolid figure contained by ſix quadrilateral figures whereof every oppoſite two are parallel.

PROP.

PROP. I. THEOR.

ONE part of a ſtraight line cannot be in a plane and another part above it. See N.

If it be poſſible, let AB, part of the ſtraight line ABC, be in a plane, and the part BC above it: And ſince the ſtraight line AB is in the plane, it can be produced in that plane: Let it be produced to D: And let any plane paſs thro' the ſtraight AD, and be turned about it until it paſs thro' the point C; and becauſe the points B, C are in this plane, the ſtraight line BC is in it [a]: Therefore there are two ſtraight lines ABC, ABD in the ſame plane that have a common ſegment AB, which is impoſſible [b]. Therefore one part, &c. Q. E. D.



[a] 7. def. 1.

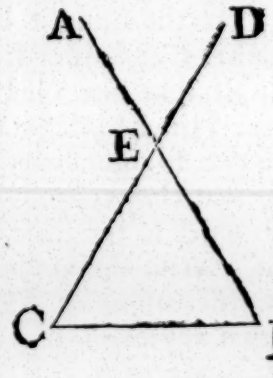
[b] Cor. 11. 1.

PROP. II. THEOR.

TWO ſtraight lines which cut one another are in one plane, and three ſtraight lines which meet one another are in one plane. See N.

Let two ſtraight lines AB, CD cut one another in E; AB, CD are one plane: And three ſtraight lines EC, CB, BE which meet one another, are in one plane.

Let any plane paſs through the ſtraight line EB, and let the plane be turned about EB, produced, if neceſſary, until it paſs through the point C: Then becauſe the points E, C are in this plane, the ſtraight line EC is in it [a]: For the ſame reaſon, the ſtraight line BC is in the ſame; and, by the hypotheſis, EB is in it: Therefore the three ſtraight lines EC, CB, BE are in one plane: But in the plane in which EC, EB are, in the ſame are [b] CD, AB: Therefore AB, CD are in one plane. Wherefore the two ſtraight lines, &c. Q. E. D.



[a] 7. def. 1.

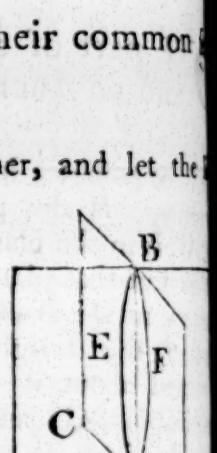
[b] 1. 11.

PROP.

PROP. III. THEOR.

See N. IF two planes cut one another, their common ſection is a ſtraight line.

Let two planes AB, BC cut one another, and let the DB be their common ſection; DB is a ſtraight line: If it be not, from the point D to B draw in the plane AB the ſtraight line DEB, and in the plane BC the ſtraight line DFB: Then two ſtraight lines DEB, DFB have the ſame extremities, and therefore include a ſpace betwixt them; which is impoſſible [a]: Therefore BD the common ſection of the planes AB, BC cannot but be a ſtraight line. Wherefore, if two planes, &c. Q. E. D.



[a] 10. Ax. 1.

PROP. IV. THEOR.

See N. IF a ſtraight line ſtand at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane which paſſes through them, that is, to the plane in which they are.

Let the ſtraight line EF ſtand at right angles to each of the ſtraight lines AB, CD in E the point of their interſection: EF is alſo at right angles to the plane paſſing thro' AB, CD.

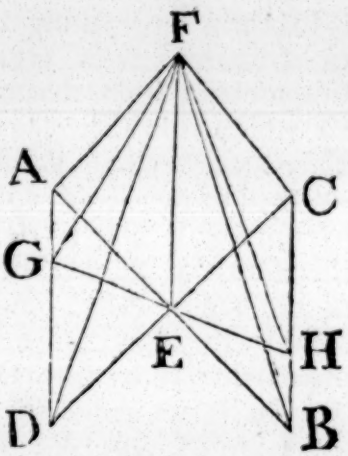
Take the ſtraight lines AE, EB, CE, ED all equal to one another; and thro' E draw, in the plane in which are AB, CD, any ſtraight line GEH; and join AD, CB; then, from any point F in EF, draw FA, FG, FD, FC, FH, FB: And becauſe the two ſtraight lines AE, ED are equal to the two BE, EC, and that they contain equal angles [a] AED, BEC, the baſe AD is equal [b] to the baſe BC, and the angle DAE to the angle EBC: And the angle AEG is equal to the angle BEH [a]; therefore the triangles AEG, BEH have two angles of one equal to two angles of the other, each to each, and the ſides adjacent to the equal angles, equal to one another; wherefore they ſhall have their other ſides equal [c]: GE is therefore equal

[a] 15. 1.

[b] 4. 1.

[c] 26. 1.

ual to EH, and AG to BH: And becauſe AE is equal to EB, Book XI.
d FE common and at right angles to them, the baſe AF is
ual [b] to the baſe FB; for the ſame reaſon, CF is equal to [b] 4. 1.
D: And becauſe AD is equal to BC, and AF to FB, the two
les FA, AD are equal to the two
B, BC, each to each; and the baſe DF
as proved equal to the baſe FC;
herefore the angle FAD is equal [d] to [d] 8. 1.
e angle FBC: Again, it was proved
at AG is equal to BH, and alſo AF
FB; FA then and AG, are equal
FB and BH, and the angle FAG
as been proved equal to the angle
BH; therefore the baſe GF is equal
to the baſe FH: Again, becauſe it
as proved, that GE is equal to EH,
nd EF is common; GE, EF are e-
ual to HE, EF; and the baſe GF
equal to the baſe FH; therefore the angle GEF is equal [d] to
e angle HEF, and conſequently each of theſe angles
a right [e] angle. Therefore FE makes right angles with [e] 10. Def. 1.
GH, that is, with any ſtraight line drawn through E in the plane
aſſing through AB, CD. In like manner, it may be proved, that
E makes right angles with every ſtraight line which meets it in
at plane. But a ſtraight line is at right angles to a plane
hen it makes right angles with every ſtraight line which meets
in that plane [f]: Therefore EF is at right angles to the plane [f] 3. def. 11.
which are AB, CD. Wherefore, if a ſtraight line, &c.
Q. E. D.



PROP. V. THEOR.

IF three ſtraight lines meet all in one point, and a See N.
ſtraight line ſtands at right angles to each of them
in that point; theſe three ſtraight lines are in one,
nd the ſame plane.

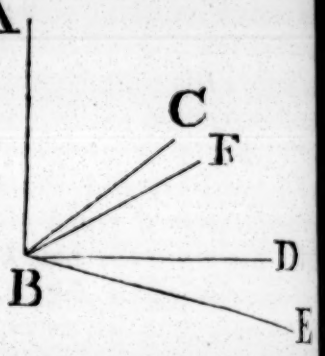
Let the ſtraight line AB ſtand at right angles to each of the
ſtraight lines BC, BD, BE, in B the point where they meet;
BC, BD, BE are in one and the ſame plane.

If not, let, if it be poſſible, BD and BE be in one plane, and
BC be above it; and let a plane paſs through AB, BC, the
common ſection of which with the plane, in which BD and BE

 are,

Book XI.

are, ſhall be a ſtraight [a] line; let this be BF: Therefore the three ſtraight lines AB, BC, BF are all in one plane, viz. that which paſſes through AB, BC: And becauſe AB ſtands at right angles to each of the ſtraight lines BD, BE, it is alſo at right angles [b] to the plane paſſing through them; and therefore makes right angles [c] with every ſtraight line meeting it in that plane; but BF which is in that plane meets it: Therefore the angle ABF is a right angle; but the angle ABC, by the hypotheſis, is alſo a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the ſame plane, which is impoſſible. Therefore the ſtraight line BC is not above the plane in which are BD and BE: Wherefore the three ſtraight lines BC, BD, BE are in one and the ſame plane. Therefore, if three ſtraight lines, &c. Q. E. D.



[a] 3. 11.

[b] 4. 11.

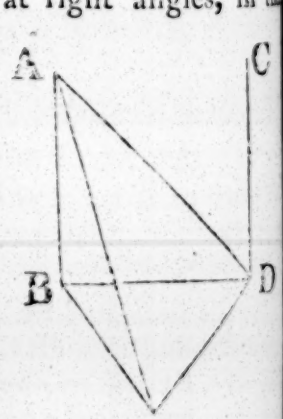
[c] 3 Def. 11.

PROP. VI. THEOR.

IF two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel to one another.

Let the ſtraight lines AB, CD be at right angles to the ſame plane; AB is parallel to CD.

Let them meet the plane in the points B, D, and draw the ſtraight line BD, to which draw DE at right angles, in the ſame plane: and make DE equal to AB, and join BE, AE, AD. Then, becauſe AB is perpendicular to the plane, it ſhall make right [a] angles with every ſtraight line which meets it, and is in that plane: But BD, BE, which are in that plane, do each of them meet AB: Therefore each of the angles ABD, ABE is a right angle: For the ſame reaſon, each of the angles CDB, CDE is a right angle: And becauſe AB is equal to DE, and BD common, the two ſides AB, BD, are equal to the two ED, DB; and they contain right angles; therefore the baſe AD is equal [b] to the baſe BE: Again, becauſe AB is equal to



[a] 3. Def. 11.

[b] 4. 1.

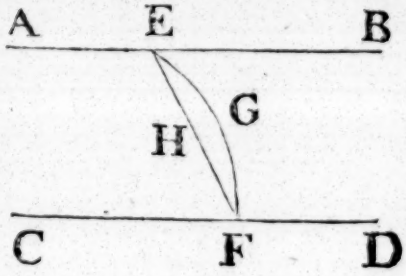
DE, and BE to AD; AB, BE are equal to ED, DA; and, Book XI.
the triangles ABE, EDA, the base AE is common; there-
re the angle ABE is equal [c] to the angle EDA: But ABE is [c] 8. 1.
right angle; therefore EDA is also a right angle, and ED
rpendicular to DA: But it is also perpendicular to each of
e two BD, DC: Wherefore ED is at right angles to each of
e three straight lines BD, DA, DC in the point in which
ey meet: Therefore these three straight lines are all in the
me plane [d]: But AB is in the plane in which are BD, DA, [d] 5. 11.
cause any three straight lines which meet one another are in [e] 2. 11.
e plane [e]: Therefore AB, BD, DC are in one plane: And
ch of the angles ABD, BDC is a right angle; therefore AB is
rallel [f] to CD. Wherefore, if two straight lines, &c. Q. E. D. [f] 28. 1.

PROP. VII. THEOR.

F two straight lines be parallel, the straight line See N.
drawn from any point in the one to any point in
e other, is in the same plane with the parallels.

Let AB, CD be parallel straight lines, and take any point in the one, and the point F in the other: The straight line which joins E and F is in the same plane with the parallels.

If not, let it be, if possible, above the plane, as EGF; and the plane ABCD in which the parallels are, draw the straight line EHF from E to F; and since EGF also is a straight line, the two straight lines EHF, EGF include a space betwixt them, which is impossible [a]. Therefore the straight line joining the points E, F is not above the plane in which the parallels AB, CD are, and is therefore in that plane. Wherefore, if two straight lines, &c. Q. E. D.



[a] 10. Ax. 1.

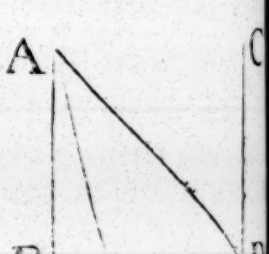
PROP. VIII. THEOR.

F two straight lines be parallel, and one of them See N.
is at right angles to a plane; the other also shall
e at right angles to the same plane.

Book XI.

Let AB, CD be two parallel ſtraight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the ſame plane.

Let AB, CD meet the plane in the points B, D, and join BD: Therefore [g] AB, CD, BD are in one plane. In the plane to which AB is at right angles, draw DE at right angles to BD, and make DE equal to AB, and join BE, AE, AD. And becauſe AB is perpendicular to the plane, it is perpendicular to every ſtraight line which meets it, and is in that plane [a]: Therefore each of the angles ABD, ABE, is a right angle: And becauſe the ſtraight line BD meets the parallel ſtraight lines AB, CD, the angles ABD, CDB are together equal [b] to two right angles: And ABD is a right angle, therefore alſo CDB is a right angle, and CD perpendicular to BD: And becauſe AB is equal to DE, and BD common, the two AB, BD, are equal to the two ED, DB, and the angle ABD is equal to the angle EDB, becauſe each of them is a right angle; therefore the baſe AD is equal [c] to the baſe BE: Again, becauſe AB is equal to DE, and BE to AD; the two AB, BE are equal to the two ED, DA; and the baſe AE is common to the triangles ABE, EDA; wherefore the angle ABE is equal [d] to the angle EDA: And ABE is a right angle; and therefore EDA is a right angle, and ED perpendicular to DA: But it is alſo perpendicular to BD; therefore ED is perpendicular [e] to the plane which paſſes through BD, DA, and ſhall [f] make right angles with every ſtraight line meeting it in that plane: But DC is in the plane paſſing through BD, DA, becauſe all three are in the plane in which are the parallels AB, CD: Wherefore ED is at right angles to DC; and therefore CD is at right angles to DE: But CD is alſo at right angles to DB; CD then is at right angles to the two ſtraight lines DE, DB in the point of their interſection D; and therefore is at right angles [e] to the plane paſſing through DE, DB, which is the ſame plane to which AB is at right angles. Therefore, if two ſtraight lines &c. Q. E. D.



[g] 7. 11.
[a] 3. Def. 11.
[b] 29. 1.
[c] 4. 1.
[d] 8. 1.
[e] 4. 11.
[f] 3. def. 11.

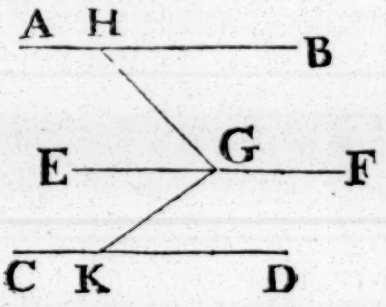
PRO

PROP. IX. THEOR.

TWO ſtraight lines which are each of them parallel to the ſame ſtraight line, and not in the ſame plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the ſame plane with it; AB ſhall be parallel to CD.

In EF take any point G, from which draw, in the plane paſſing through EF, AB, the ſtraight line GH at right angles to EF; and in the plane paſſing through EF, CD, draw GK at right angles to the ſame EF. And becauſe EF is perpendicular both to GH and GK, EF is perpendicular [a] to the plane HGK paſſing through them: And EF is parallel to AB; therefore AB is at right angles [b] to the plane HGK. For the ſame reaſon, CD is likewiſe at right angles to the plane HGK. Therefore AB, CD are each of them at right angles to the plane HGK. But if two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel [c] to one another. Therefore AB is parallel to CD. Wherefore two ſtraight lines, &c. Q. E. D.



[a] 4. 11.

[b] 8. 11.

[c] 6. 11.

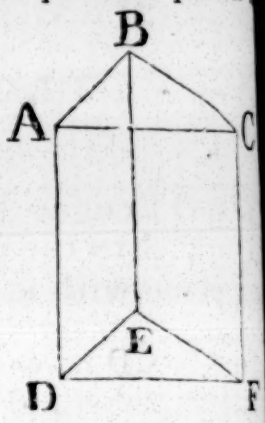
PROP. X. THEOR.

IF two ſtraight lines meeting one another be parallel to two others that meet one another, and are not in the ſame plane with the firſt two; the firſt two and the other two ſhall contain equal angles.

Let the two ſtraight lines AB, BC which meet one another be parallel to the two ſtraight lines DE, EF that meet one another, and are not in the ſame plane with AB, BC. The angle ABC is equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another; and join

Book XI. AD, CF, BE, AC, DF; Becauſe BA is equal and parallel
[a] 33. 1. ED, therefore AD is [a] both equal and parallel to BE. For the ſame reaſon, CF is equal and parallel to BE. Therefore AD and CF are each of them equal and parallel to BE. But ſtraight lines that are parallel to the ſame ſtraight line, and not in the ſame plane with it,
[b] 9. 11. are parallel [b] to one another. Therefore
[c] 1. Ax. 1. AD is parallel to CF; and it is equal [c] to it, and AC, DF join them towards the ſame parts; and therefore [a] AC is equal and parallel to DF. And becauſe AB, BC are equal to DE, EF, and the baſe AC to the
[d] 8. 1. baſe DF; the angle ABC is equal [d] to the angle DEF. Therefore if two ſtraight lines, &c. Q. E. D.

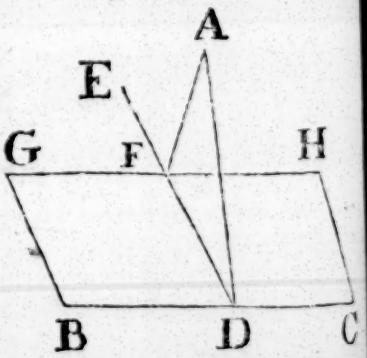


PROP. XI. PROB.

TO draw a ſtraight line perpendicular to a plane from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a ſtraight line perpendicular to the plane BH.

In the plane draw any ſtraight line BC, and from the point
[a] 12. 1 A draw [a] AD perpendicular to BC. If then AD be alſo perpendicular to the plane BH, the thing required is already done; but if it be not, from the
[b] 11. 1. point D draw [b] in the plane BH, the ſtraight line DE at right angles to BC; and from the point A draw AF perpendicular to
[c] 31. 1. DE; and thro' F draw [c] GH parallel to BC: And becauſe BC is at right angles to ED and DA,
[d] 4. 11. BC is at right angles [d] to the plane paſſing through ED, DA. And GH is parallel to BC; but if two ſtraight lines be parallel, one of which is at right angles to
[e] 8. 11. a plane, the other ſhall be at right [e] angles to the ſame plane; wherefore GH is at right angles to the plane through ED, DA,
[f] 3. Def. 11. and is perpendicular [f] to every ſtraight line meeting it in that plane. But AF, which is in the plane through ED, DA, meets

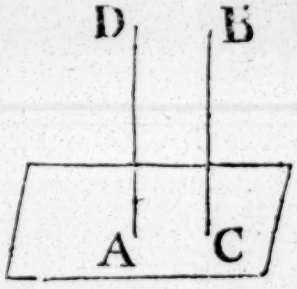


; Therefore GH is perpendicular to AF, and consequently F is perpendicular to GH; and AF is perpendicular to DE; erefore AF is perpendicular to each of the ſtraight lines GH, E. But if a ſtraight line ſtands at right angles to each of vo ſtraight lines in the point of their interſection, it ſhall alſo at right angles to the plane paſſing through them. But the ane paſſing through ED, GH is the plane BH; therefore AF perpendicular to the plane BH. Therefore from the given oint A above the plane BH, the ſtraight line AF is drawn erpendicular to that plane. Which was to be done. Book XI.

PROP. XII. PROB.

TO erect a ſtraight line at right angles to a given plane, from a point given in the plane.

Let A be the point given in the plane; it is required to erect ſtraight line from the point A at right ngles to the plane.



From any point B above the plane raw [a] BC perpendicular to it; and om A draw [b] AD parallel to BC. Beauſe therefore AD, CB are two pallel ſtraight lines, and one of them C is at right angles to the given plane, he other AD is alſo at right angles to [c]. Therefore a ſtraight line has been erected at right angles to given plane from a point given in it. Which was to be done.

[a] 11. 11.
[b] 31 1.
[c] 8. 11.

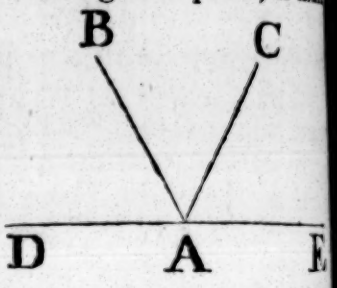
PROP. XIII. THEOR.

FROM the ſame point in a given plane, there cannot be two ſtraight lines at right angles to he plane, upon the ſame ſide of it: And there can e but one perpendicular to a plane from a point bove the plane.

For, if it be poſſible, let the two ſtraight lines AB, AC be at ght angles to a given plane from the ſame point A in the plane, nd upon the ſame ſide of it; and let a plane paſs through BA, AC; the common ſection of this with the given plane is a ſtraight

Book XI. [a] line passing through A: Let DAE be their common section: Therefore the straight lines AB, AC, DAE are in one plane:

[a] 3. 11. And because CA is at right angles to the given plane, it shall make right angles with every straight line meeting it in that plane. But DAE, which is in that plane, meets CA; therefore CAE is a right angle. For the same reason BAE is a right angle. Wherefore the angle CAE is equal to the angle BAE; and they are in one plane, which is impossible. Also, from a point above a plane, there can be but one perpendicular to that plane; for if there could be two,



[b] 6. 11. they would be parallel [b] to one another, which is absurd. Therefore, from the same point, &c. Q. E. D.

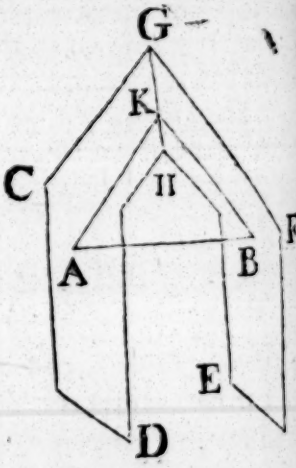
PROP. XIV. THEOR.

PLANES to which the same straight line is perpendicular, are parallel to one another.

Let the straight line AB be perpendicular to each of the planes CD, EF; these planes are parallel to one another.

If not, they shall meet one another when produced; let them meet; their common section shall be a straight line GH, in which take any point K, and join AK, BK: Then, because AB is perpendicular to

[a] 3. def. 11. the plane EF, it is perpendicular [a] to the straight line BK which is in that plane. Therefore ABK is a right angle. For the same reason, BAK is a right angle; wherefore the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is impossible [b]: Therefore the planes CD, EF though produced do not meet one another; that is, they are parallel [c]. Therefore planes, &c. Q. E. D.



[b] 17. 1.

[c] 8. Def. 11.

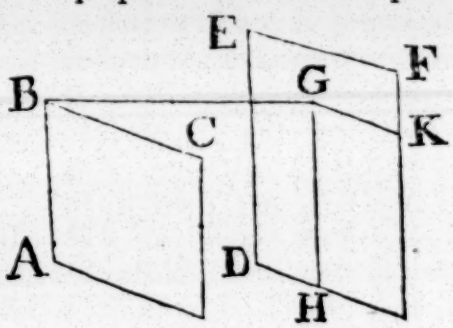
PROP

PROP. XV. THEOR.

F two ſtraight lines meeting one another, be pa- See N.
rallel to two ſtraight lines which meet one ano-
er, but are not in the ſame plane with the firſt
vo; the plane which paſſes through theſe is parallel
the plane paſſing through the others.

Let AB, BC, two ſtraight lines meeting one another, be pa-
lel to **DE**, EF that meet one another, but are not in the
me plane with **AB**, BC: The planes through AB, BC, and
E, EF ſhall not meet though produced.

From the point B draw BG perpendicular [a] to the plane [a] 11. 11.
hich paſſes through DE, EF, and let it meet that plane in
; and through G draw GH parallel [b] to ED, and GK pa- [b] 31. 1.
lel to EF: And becauſe BG is perpendicular to the plane
rough DE, EF, it ſhall
ake right angles with every
aight line meeting it in that
ane [c]: But the ſtraight lines
H, GK in that plane meet
: Therefore each of the an-
es BGH, BGK is a right
gle: And becauſe BA is
rallel [d] to GH (for each of
em is parallel to DE, and
ey are not both in the ſame plane with it) the angles GBA,
GH are together equal [e] to two right angles: And BGH is a [e] 29. 1.
ght angle, therefore alſo GBA is a right angle, and GB per-
ndicular to BA: For the ſame reaſon, GB is perpendicular
BC; Since therefore the ſtraight line GB ſtands at right an-
es to the two ſtraight lines BA, BC, that cut one another in
; GB is perpendicular [f] to the plane through BA, BC: And [f] 4. 11.
is perpendicular to the plane through DE, EF; therefore BG
perpendicular to each of the planes through AB, BC and DE,
: But planes to which the ſame ſtraight line is perpendicular,
parallel [g] to one another: Therefore the plane through AB, [g] 14. 11.
C is parallel to the plane through DE, EF. Wherefore, if
o ſtraight lines, &c. Q. E. D.



[c] 3. Def. 11.

[d] 9. 11.

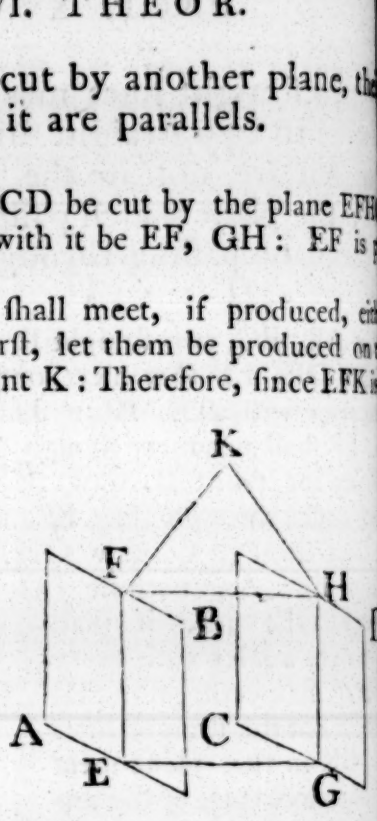
PROP.

PROP. XVI. THEOR.

See N. IF two parallel planes be cut by another plane, the common ſections with it are parallels.

Let the parallel planes AB, CD be cut by the plane EFH and let their common ſections with it be EF, GH: EF is parallel to GH.

For, if it is not, EF, GH ſhall meet, if produced, either on the ſide of FH, or EG: Firſt, let them be produced on the ſide of FH, and meet in the point K: Therefore, ſince EFK is in the plane AB, every point in EFK is in that plane; and K is a point in EFK; therefore K is in the plane AB: For the ſame reaſon K is alſo in the plane CD: Wherefore the planes AB, CD produced meet one another; but they do not meet, ſince they are parallel by the hypotheſis: Therefore the ſtraight lines EF, GH do not meet when produced on the ſide of FH: In the ſame manner it may be proved, that EF, GH do not meet when produced on the ſide of EG: But ſtraight lines which are in the ſame plane and do not meet, though produced either way, are parallel: Therefore EF is parallel to GH. Wherefore, if two parallel planes, &c. Q. E. D.



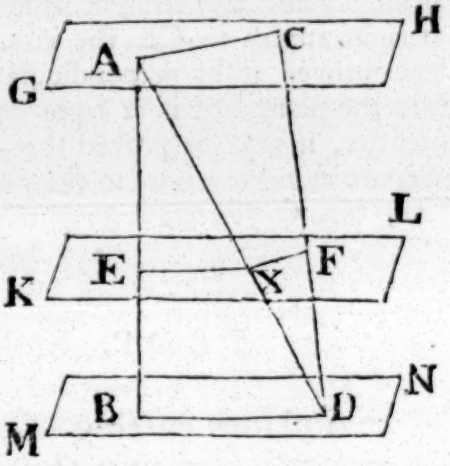
PROP. XVII. THEOR.

IF two ſtraight lines be cut by parallel planes, they ſhall be cut in the ſame ratio.

Let the ſtraight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D: As AE to EB, ſo is CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF: Becauſe the two parallel planes KL, MN are cut by the plane EBDX, the common ſections

, BD are parallel [a]. For the ſame reaſon, becauſe the two Book XI.
allel planes GH, KL are
by the plane AXFC, the
mmon ſections AC, XF are
allel: And becauſe EX is
allel to BD, a ſide of the tri-
gle ABD, as AE to EB, ſo
AX to XD. Again, be-
ſe XF is parallel to AC, a
of the triangle ADC, as
X to XD, ſo is CF to FD:
d it was proved that AX is
XD, as AE to EB: There-
[c] as AE to EB, ſo is CF
FD. Wherefore, if two
ight lines, &c. Q. E. D.



[a] 16. 11.

[b] 2. 6.

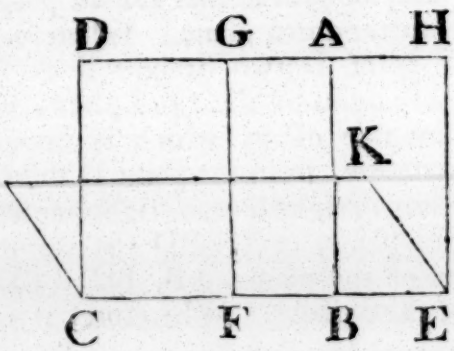
[c] 11. 5.

PROP. XVIII. THEOR.

F a ſtraight line be at right angles to a plane, every plane which paſſes through it ſhall be at right angles to that plane.

Let the ſtraight line AB be at right angles to the plane CK: y plane which paſſes through AB ſhall be at right angles to plane CK.

et any plane DE paſs through AB, and let CE be the mmon ſection of the planes DE, CK; take any point F in from which draw FG in plane DE at right angles CE: And becauſe AB is pendicular to the plane , therefore it is alſo per-dicular to every ſtraight in that plane meeting it [a]; d conſequently it is perpen-ular to CE: Wherefore F is a right angle; but B is likewiſe a right angle; efore AB is parallel [b] to FG. And AB is at right angles to plane CK; therefore FG is alſo at right angles to the ſame ne [c]. But one plane is at right angles to another plane when ſtraight lines drawn in one of the planes, at right angles



[a] 3. def. 11.

[b] 28. 1.

[c] 8. 11.

to

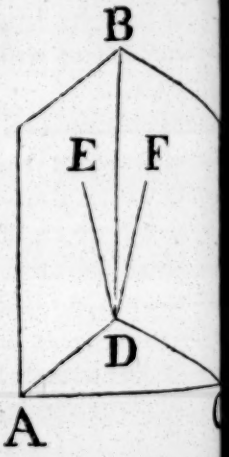
Book XI. to their common ſection, are alſo at right angles to the o
plane [d]; and any ſtraight line FG in the plane DE, which
[d] 4. Def. 11. at right angles to CE the common ſection of the planes, h
been proved to be perpendicular to the other plane CK; the
fore the plane DE is at right angles to the plane CK. In l
manner, it may be proved that all the planes which paſs thro
AB are at right angles to the plane CK. Therefore, if a ſtrai
line, &c. Q. E. D.

PROP. XIX. THEOR.

IF two planes cutting one another be each of th
perpendicular to a third plane; their common ſ
tion ſhall be perpendicular to the ſame plane.

Let the two planes AB, BC be each of them perpendicu
to a third plane, and let BD be the common ſection of the t
two; BD is perpendicular to the third plane.

If it be not, from the point D draw, in the plane AB, t
ſtraight line DE at right angles to AD the common ſecti
of the plane AB with the third plane; and in the plane
draw DF at right angles to CD the common ſection of t
plane BC with the third plane. And be-
cauſe the plane AB is perpendicular to
the third plane, and DE is drawn in the
plane AB at right angles to AD their
common ſection, DE is perpendicular to
[a] 4. Def. 11. the third plane [a]. In the ſame manner,
it may be proved that DF is perpendicu-
lar to the third plane. Wherefore, from
the point D two ſtraight lines ſtand at
right angles to the third plane, upon the
[b] 13. 11. ſame ſide of it, which is impoſſible [b].
Therefore, from the point D there cannot
be any ſtraight line at right angles to the
third plane, except BD the common ſec-
tion of the planes AB, BC. BD therefore is perpendicular
the third plane. Wherefore, if two planes, &c. Q. E D.



PRO

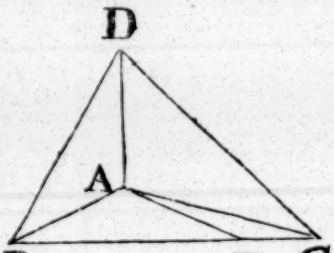
PROP. XX. THEOR.

Book XI.

a ſolid angle be contained by three plane angles, ny two of them are greater than the third. See N.

et the ſolid angle at A be contained by the three plane an- BAC, CAD, DAB. Any two of them are greater than third.

f the angles BAC, CAD, DAB be all equal, it is evident any two of them are greater than the third. But if they not, let BAC be that angle which is not leſs than either he other two, and is greater than one of them DAB; and he point A in the ſtraight line AB, make in the plane ch paſſes through BA, AC, the angle BAE equal [a] to the le DAB; and make AE equal to AD, and through E w BEC cutting AB, AC in the nts B, C, and join DB, DC. And auſe DA is equal to AE, and AB ommon, the two DA, AB are e- l to the two EA, AB, and the le DAB is equal to the angle EAB. erefore the baſe DB is equal [b] to baſe BE. And becauſe BD, DC greater [c] than CB, and one of them has been proved equal to BE a part of CB, therefore the er DC is greater than the remaining part EC. And becauſe A is equal to AE, and AC common, but the baſe DC greater n the baſe EC; therefore the angle DAC is greater [d] than angle EAC; and, by the conſtruction, the angle DAB is al to the angle BAE; wherefore the angles DAB, DAC are ether greater than BAE, EAC, that is, than the angle BAC. BAC is not leſs than either of the angles DAB, DAC; erefore BAC with either of them is greater than the other. herefore, if a ſolid angle, &c. Q. E. D.



[a] 23. 1.

[b] 4. 1.

[c] 20. 1.

[d] 25. 1.

PROP. XXI. THEOR.

VERY ſolid angle is contained by plane angles which together are leſs than four right angles.

irſt, Let the ſolid angle at A be contained by three plane les BAC, CAD, DAB. Theſe three together are leſs than r right angles.

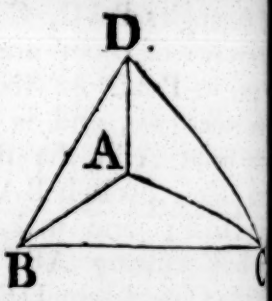
Take

Book XI.

Take in each of the ſtraight lines AB, AC, AD any points B, C, D, and join BC, CD, DB: Then, becauſe the ſolid angle at B is contained by the three plain angles CBA, ABD, DBC, any two of them are greater [a] than the third; therefore the angles CBA, ABD are greater than the angle DBC. For the ſame reaſon, the angles BCA, ACD are greater than the angle DCB; and the angles CDA, ADB greater than BDC: Wherefore the ſix angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, BCD, CDB: But the three angles DBC, BCD, CDB are equal to two right angles [b]: Therefore the ſix angles CBA, ABD: BCA, ACD, CDA, ADB are greater than two right angles: And becauſe the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles, therefore the nine angles of theſe three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD, are equal to ſix right angles: Of theſe the ſix angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles: Therefore the remaining three angles BAC, DAC, BAD, which contain the ſolid angle at A, are leſs than four right angles.

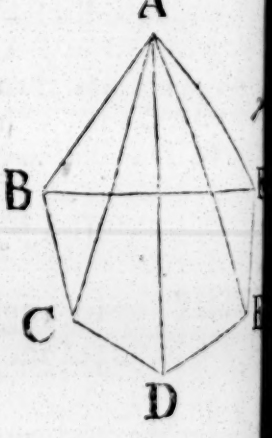
[a] 20. 11.

[b] 32. 1.



Next, Let the ſolid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; theſe together are leſs than four right angles.

Let the planes in which the angles are, be cut by a plane, and let the common ſections of it with thoſe planes be BC, CD, DE, EF, FB: And becauſe the ſolid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater [a] than the third, the angles CBA, ABF are greater than the angle FBC: For the ſame reaſon, the two plane angles at each of the points C, D, E, F, viz. the angles which are at the baſes of the triangles having the common vertex A, are greater than the third angle at the ſame point, which is one of the angles of the polygon BCDEF: Therefore all the angles at the baſes of the triangles are to-



greater than all the angles of the polygon: And becauſe Book XI.
he angles of the triangles are together equal to twice as
y right angles as there are triangles [b]; that is, as there are [b] 32. 1.
in the polygon BCDEF; and that all the angles of the
gon together with four right angles are likewiſe equal to
e as many right angles as there are ſides in the polygon [c]; [c] 1. Cor.
efore all the angles of the triangles are equal to all the an- 32. 1.
of the polygon together with four right angles. But all
ngles at the baſes of the triangles are greater than all the
es of the polygon, as has been proved. Wherefore the re-
ing angles of the triangles, viz. thoſe at the vertex, which
ain the ſolid angle at A, are leſs than four right angles.
efore every ſolid angle, &c, Q. E. D.

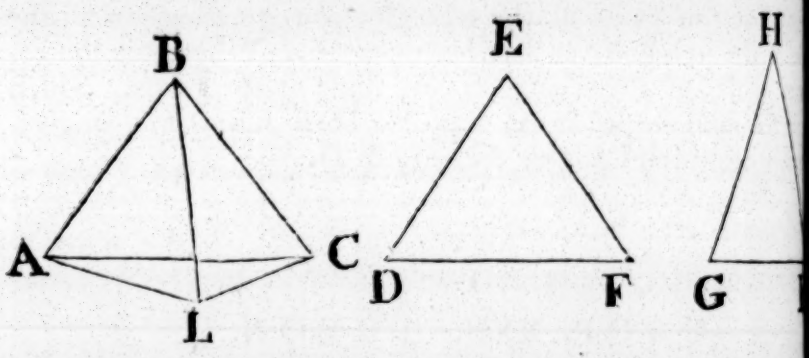
PROP. XXII. THEOR.

every two of three plane angles be greater than See N.
he third, and if the ſtraight lines which contain
m be all equal; a triangle may be made of the
ight lines that join the extremities of thoſe equal
ight lines.

et ABC, DEF, GHK be three plane angles, whereof e-
two are greater than the third, and are contained by the
l ſtraight lines AB, BC, DE, EF, GH, HK; if their
mities be joined by the ſtraight lines AC, DF, GK, a
gle may be made of three ſtraight lines equal to AC, DF,
; that is, every two of them are together greater than the

the angles at B, E, H are equal; AC, DF, GK are al-
ual [a], and any two of them greater than the third: But [a] 4. 1.
e angles are not all equal, let the angle ABC be not leſs
either of the two at E, H; therefore the ſtraight line
is not leſs than either of the other two DF, GK [b]; and [b] 4. or 24.
plain that AC together with either of the other two muſt 1.
reater than the third: Alſo DF with GK are greater than
: For, at the point B in the ſtraight line AB make [c] the [c] 23. 1.
angle

Book XI. angle ABL equal to the angle GHK, and make BL equal one of the ſtraight lines AB, BC, DE, EF, GH, HK, join AL, LC: Then becauſe AB, BL are equal to GH, and the angle ABL to the angle GHK, the baſe AL is to the baſe GK: And becauſe the angles at E, H are than the angle ABC, of which the angle at H is equal to A therefore the remaining angle at E is greater than the angle L



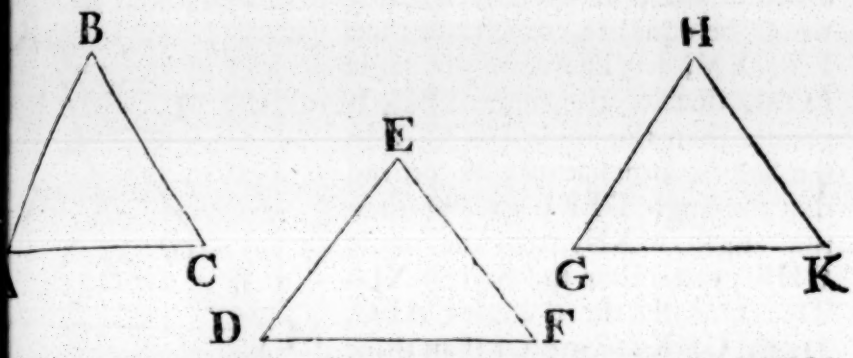
And becauſe the two ſides LB, BC are equal to the two EF, and that the angle DEF is greater than the angle L
[d] 24. 1. the baſe DF is greater [d] than the baſe LC: And it has proved that GK is equal to AL; therefore DF and GK
[e] 20. 1. greater than AL and LC: But AL and LC are greater [e] AC; much more then are DF and GK greater than AC. Wherefore every two of the ſtraight lines AC, DF, GK are
[f] 22. 1. than the third; and therefore a triangle may be made [f] the of which ſhall be equal to AC, DF, GK. Q. E. D.

PROP. XXIII. PROB.

See N. TO make a ſolid angle which ſhall be contained three given plane angles, any two of them ing greater than the third, and all three toge leſs than four right angles.

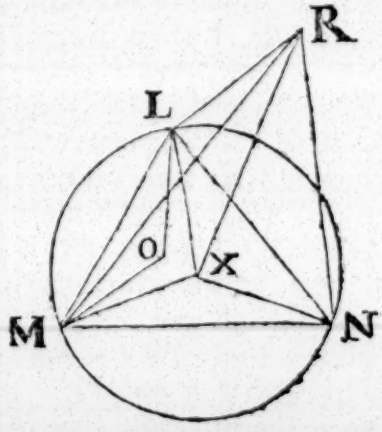
Let the three given plane angles be ABC, DEF, GHK two of which are greater than the third, and all of them gether leſs than four right angles. It is required, to m ſolid angle contained by three plane angles equal to ABC, GHK, each to each.

From the ſtraight lines containing the angles, cut off AB, C, DE, EF, GH, HK all equal to one another; and join C, DF, GK: Then a triangle may be made [a] of three ſtraight [a] 22. 11. Book. XI.



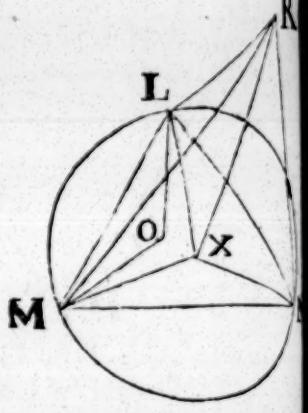
nes equal to AC, DF, GK. Let this be the triangle LMN [b], [b] 22. 1.
that AC be equal to LM, DF to MN, and GK to LN; and
out the triangle LMN deſcribe [c] a circle, and find its center [c] 5. 4.
, which will either be within the triangle, or in one of its
les, or without it.

Firſt, Let the center X be within the triangle, and join X, MX, NX: AB is greater than LX: If not, AB muſt either be equal to, or leſs than LX; firſt, let it be equal: Then ecauſe AB is equal to LX, and that AB is alſo equal to BC, d LX to XM, AB and BC are equal to LX and XM, each each; and the baſe AC is, by conſtruction, equal to the ſe LM; wherefore the angle ABC is equal to the angle XM [d]: For the ſame reaſon the angle DEF is equal to the [d] 8. 1.
gle MXN, and the angle GHK the angle NXL: Therefore the ree angles ABC, DEF, GHK e equal to the three angles LXM, XN, NXL: But the three angles XM, MXN, NXL are equal to ur right angles [e]; therefore alſo e three angles ABC, DEF, GHK e equal to four right angles: But, the hypotheſis, they are leſs an four right angles; which is ſurd: Therefore AB is not equal LX: But neither can AB be s than LX: For, if poſſible, let it be leſs, and upon the aight line LM, on the ſide of it on which is the center X, ſcribe the triangle LOM, the ſides LO, OM of which are ual to AB, BC; and becauſe the baſe LM is equal to the

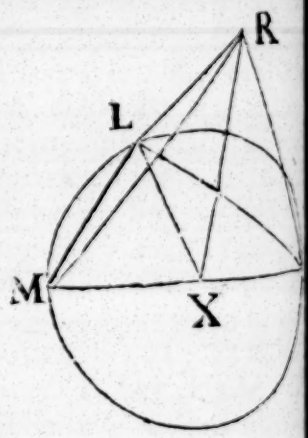


[e] 2. Cor. 15. 1.

Book XI. base AC, the angle LOM is equal to the angle ABC[d]: A
d 8. 1. AB, that is, LO, by the hypothesis, is less than LX; wh
fore LO, OM fall within the triangle LXM; for, if they
upon its sides, or without it, they
would be equal to, or greater than
f 21. 1 LX, XM, [f]: Therefore the angle
LOM, that is, the angle ABC, is
greater than the angle LXM[f]: In
the same manner it may be proved
that the angle DEF is greater than
the angle MXN, and the angle
GHK greater than the angle NXL:
Therefore the three angles ABC,
DEF, GHK are greater than three
angles LXM, MXN, NXL; that
is, than four right angles: But the
same angles ABC, DEF, GHK are less than four right ang
which is absurd: Therefore AB is not less than LX, and it
been proved that it is not equal to LX; wherefore AB is gre
than LX.



Next, Let the center X of the circle fall in one of the
of the triangle, viz. in MN, and
join XL: In this case also AB is
greater than LX If not, AB is
either equal to LX or less than it:
First, let it be equal to LX: There-
fore AB and BC, that is, DE and
EF, are equal to MX and XL, that
is, to MN: But, by the construction,
MN is equal to DF; therefore DE
EF are equal to DF, which is im
† 20. 1. possible †: Wherefore AB is not e
qual to LX; nor is it less; for then,
much more, an absurdity would
follow: Therefore AB is greater than LX.

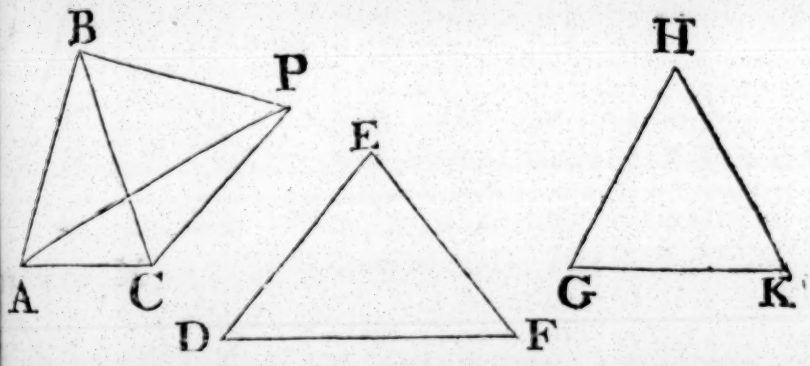


But, let the center X of the circle fall without the tria
LMN, and join LX, MX, NX. In this case likewise A
greater than LX: If not, it is either equal to, or less than
First, let it be equal; it may be proved, in the same man
as in the first case, that the angle ABC is equal to the a
MXN, and GHK to LXN; therefore the whole angle M
is equal to the two angles ABC, GHK: But ABC and G
are together greater than the angle DEF; therefore
the angle MXN is greater than DEF. And because

F are equal to MX, XN, and the base DF to the base N, the angle MXN is equal [d] to the angle DEF. And it has en proved that it is greater than DEF, which is absurd. herefore AB is not equal to LX. Nor yet is it less; for then, has been proved in the first case, the angle ABC is greater an the angle MXL, and the angle GHK greater than the gle LXN. At the point B in the straight line CB make the gle CBP equal to the angle GHK, and make BP equal to

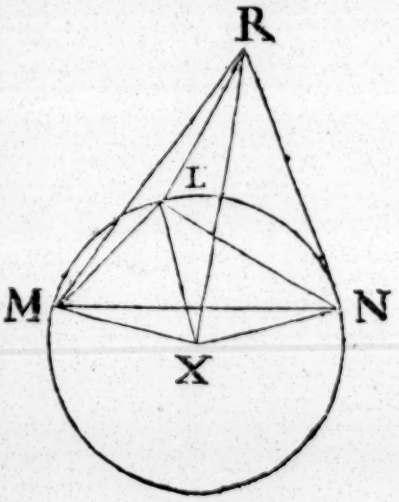
Book XI.

[d] 8. 1.



K, and join CP, AP. And because CB is equal to GH; B, BP are equal to GH, HK, each to each, and they con- n equal angles; wherefore the base CP is equal to the base K, that is, to LN. And in the isosceles triangles ABC, MXL, cause the angle ABC is greater than the angle MXL, there- e the angle MLX at the base is greater [g] than the angle CB at the base. For the same reason, because the angle GHK, CBP, is greater than the angle XN, the angle XLN is greater an the angle BCP. Therefore the ole angle MLN is greater than whole angle ACP. And because L, LN are equal to AC, CP, h to each, but the angle MLN ater than the angle ACP, the e MN is greater [h] than the base P. And MN is equal to DF; erefore also DF is greater than P. Again, because DE, EF are al to AB, BP, but the base DF ater than the base AP, the an- DEF is greater [k] than the angle BP. And ABP is equal to the two angles ABC, CBP, that to the two angles ABC, GHK; therefore the angle DEF is ater than the two angles ABC, GHK; but it is also less an these; which is impossible. Therefore AB is not less than

[g] 32. 1.

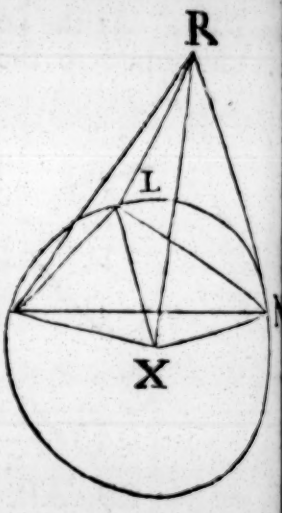


[h] 24. 1

[k] 25. 1.

Book XI. LX; and it has been proved that it is not equal to it; therefore AR is greater than LX.

[a] 12. 11. From the point X erect [a] XR at right angles to the plane of the circle LMN. And because it has been proved in all the cases, that AB is greater than LX, find a square equal to the excess of the square of AB above the square of LX, and make RX equal to its side, and join RL, RM, RN. Because RX is perpendicular to the plane of the circle LMN, it



[b] 3. def. 11. is [b] perpendicular to each of the straight lines LX, MX, NX And because LX is equal to MX, and XR common, and at right angles to each of them, the base RL is equal to the base RM For the same reason, RN is equal to each of the two RL, RM. Therefore the three straight lines RL, RM, RN are all equal. And because the square of XR is equal to the excess of the square of AB above the square of LX; therefore the square of AB is equal to the squares

[c] 47. 1. LX, XR. But the square of RL is equal [c] to the same square because LXR is a right angle. Therefore the square of AB is equal to the square of RL, and the straight line AB to RL. But each of the straight lines BC, DE, EF, GH, HK is equal to AB, and each of the two RM, RN is equal to RL. Wherefore AB, BC, DE, EF, GH, HK are each of them equal to each of the straight lines RL, RM, N. And because RL, RM, are equal to AB, BC, and the base LM to the base AC,

[d] 8. 1. the angle LRM is equal [d] to the angle ABC For the same reason, the angle MRN is equal to the angle DEF, and NRL to GHK. Therefore there is made a solid angle at R, which is contained by three plane angles LRM, MRN, NRL, which are equal to the three given plane angles ABC, DEF, GHK each to each. Which was to be done.

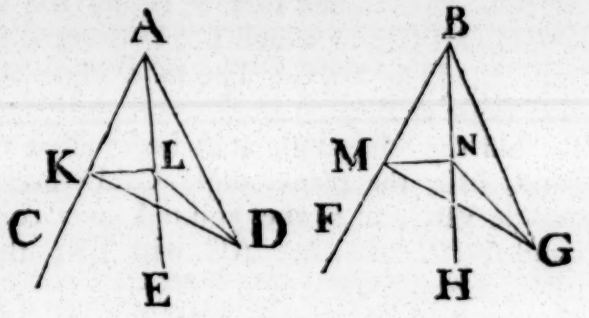
PRO

PROP. A. THEOR.

IF each of two ſolid angles be contained by three plane angles equal to one another, each to each; the planes in which the equal angles are, have the ſame inclination to one another. See N.

Let there be two ſolid angles at the points A, B; and let the angle at A be contained by the three plane angles CAD, CAE, EAD; and the angle at B by the three plane angles FBG, FBH, HBG; of which the angle CAD is equal to the angle FBG, and CAE to FBH, and EAD to HBG. The planes in which the equal angles are, have the ſame inclination to one another.

In the ſtraight line AC take any point K, and in the plane CAD from K draw the ſtraight line KD at right angles to AC, and in the plane CAE the ſtraight line KL at right angles to the ſame AC: Therefore the angle DKL is the inclination [a] of the plane CAD to the plane CAE: In BF take BM equal to AK, and from the point M draw, in the planes FBG, FBH, the ſtraight lines MG, MN at right angles to BF; therefore the angle GMN is the inclination [a] of the plane FBG to the plane FBH: Join KD, NG; and becauſe in the triangles KAD, MBG, the angles KAD, MBG are equal, as alſo the right angles AKD, BMG, and that the ſides AK, BM, adjacent to the equal angles, are equal to one another, therefore KD is equal [b] to MG, and AD to BG: For the ſame reaſon, in the triangles KAL, MBN, KL is equal to MN, and AL to BN: And in the triangles LAD, NBG, LA, AD are equal to NB, BG, and they contain equal angles; therefore the baſe LD is equal [c] to the baſe NG. Laſtly, in the triangles KLD, MNG, the ſides DK, KL are equal to GM, MN, and the baſe LD to the baſe NG; therefore the angle DKL is equal [d] to the angle GMN: But the angle DKL is the inclination of the plane CAD to the plane CAE, and the angle GMN is the inclination



[a] 6. def. 11.

[b] 26. 1.

[c] 4. 1.

[d] 8. 1.

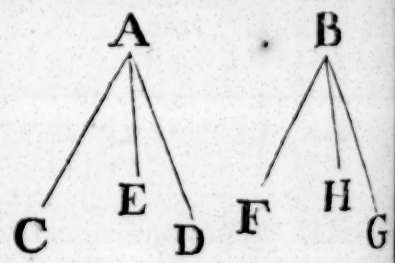
Book XI. tion of the plane FBG to the plane FBH, which planes hav
[a] 7. def. 11. therefore the ſame inclination [a] to one another: And in th
ſame manner it may be demonſtrated, that the other planes i
which the equal angles are, have the ſame inclination to on
another. Therefore, if two ſolid angles, &c. Q. E. D.

PROP. B. THEOR.

See N. IF two ſolid angles be contained, each by three plan
angles which are equal to one another, each t
each, and alike ſituated; theſe ſolid angles are equ
to one another.

Let there be two ſolid angles at A and B, of which the ſo
lid angle at A is contained by the three plane angles CA
CAE, EAD; and that at B, by the three plane angles FBG
FBH, HBG; of which CAD is equal to FBG; CAE to FBH
and EAD to HBG: The ſolid angle at A, is equal to the ſo
lid angle at B.

Let the ſolid angle at A be applied to the ſolid angle at B
and, firſt, the plane angle CAD being applied to the plan
angle FBG, ſo as the point A may coincide with the point B
and the ſtraight line AC with BF; then AD coincides wi
BG, becauſe the angle CAD is equal to the angle FBG: And becauſe the inclination of the plane CAE to the plane
[a] A. 11. CAD is equal [a] to the inclination of the plane FBH to the plane FBG, the plane CAE coincides with the plane FBH, becauſe the planes CAD, FBG coincide with one another: An
becauſe the ſtraight lines AC, BF coincide, and that the angl
CAE is equal to the angle FBH; therefore AE coincides wit
BH: And AD coincides with BG; wherefore the plane EA
coincides with the plane HBG: Therefore the ſolid angle A
coincides with the ſolid angle B, and conſequently they are
[b] 8. A. 1. qual [b] to one another. Q. E. D.



PROP

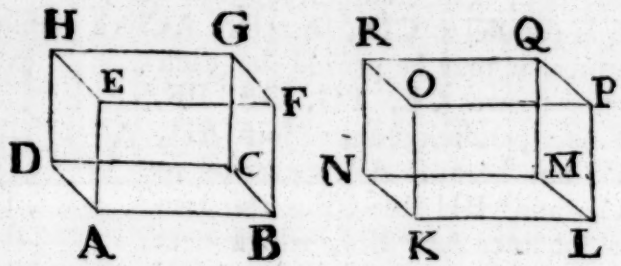
PROP. C. THEOR.

SOLID figures contained by the ſame number of equal and ſimilar planes alike ſituated, and having none of their ſolid angles contained by more than three plane angles; are equal and ſimilar to one another. See N.

Let AG, KQ be two ſolid figures contained by the ſame number of ſimilar and equal planes, alike ſituated, viz. let the plane AC be ſimilar and equal to the plane KM; the plane AF to KP; BG to LQ; GD to QN; DE to NO; and laſtly, FH ſimilar and equal to PR: The ſolid figure AG is equal and ſimilar to the ſolid figure KQ.

Becauſe the ſolid angle at A is contained by the three plane angles BAD, BAE, EAD, which, by the hypotheſis, are equal to the plane angles LKN, LKO, OK which contain the ſolid angle at K, each to each; therefore the ſolid angle at A is equal [a] to the ſolid angle at K: In the ſame manner, the other ſolid angles of the figures are equal to one another. If then the ſolid figure AG be applied to the ſolid figure KQ, firſt, the plane figure AC being applied to the plane figure KM; the ſtraight line AB coinciding with KL, the figure AC muſt coincide with the figure KM, becauſe they are equal and ſimilar: Therefore the ſtraight lines AD, DC, CB coincide with KN, NM, ML, each with each; and the points A, D, C, B with the points K, N, M, L: And the ſolid angle at A coincides [a] with the ſolid angle at K; wherefore the plane AF coincides with the plane KP, and the figure AF with the figure KP, becauſe they are equal and ſimilar to one another: Therefore the ſtraight lines AE, EF, FB coincide with KO, OP, PL; and the points E, F with the points O, P. In the ſame manner, the figure AH coincides with the figure KR, and the ſtraight line DH with NR, and the point H with the point R: And becauſe the ſolid angle at B is equal to the ſolid angle at L, it may be proved in the ſame manner, that the figure BG coincides with

[a] B. 11.



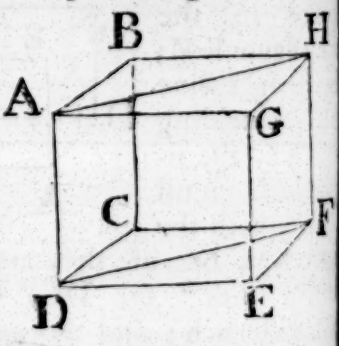
Book XI. the figure LQ, and the ſtraight line CG with MQ, and the point G with the point Q: Since therefore all the planes and ſides of the ſolid figure AG coincide with the planes and ſides of the ſolid figure KQ, AG is equal and ſimilar to KQ. And in the ſame manner, any other ſolid figures whatever contained by the ſame number of equal and ſimilar planes, alike ſituated, and having none of their ſolid angles contained by more than three plane angles, may be proved to be equal and ſimilar to one another. Q. E. D.

PROP. XXIV. THEOR.

See N. IF a ſolid be contained by ſix planes, two and two of which are parallel; the oppoſite planes are ſimilar and equal parallelograms.

Let the ſolid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE: Its oppoſite planes are ſimilar and equal parallelograms.

Becauſe the two parallel planes, BG, CE are cut by the
[a] 16. 11. plane AC, their common ſections AB, CD are parallel [a]. Again, becauſe the two parallel planes BF, AE are cut by the plane AC, their common ſections AD, BC are parallel [a]: And AB is parallel to CD; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram: Join AH, DF; and becauſe AB is parallel to DC, and BH to CF; the two ſtraight lines AB, BH, which meet one another, are parallel to DC and CF which meet one another, and are not in the ſame plane with the other two; wherefore they con-



[b] 10. 11. tain equal angles [b]; the angle ABH is therefore equal to the angle DCF: And becauſe AB, BH are equal to DC, CF, and the angle ABH equal to the angle DCF, therefore the baſe
[c] 4. 1. AH is equal [c] to the baſe DF, and the triangle ABH to the tri-
[d] 34. 1. angle DCF. And the parallelogram BG is double [d] of the triangle ABH, and the parallelogram CE double of the triangle DCF; therefore the parallelogram BG is equal and ſimilar to the parallelogram CE. In the ſame manner, it may be proved, that the parallelogram AC is equal and ſimila

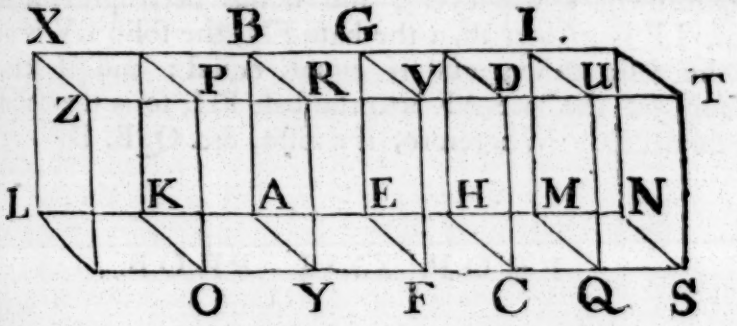
lar to the parallelogram GF, and the parallelogram AE to . Therefore, if a ſolid, &c. Q. E. D. Book XI.

PROP. XXV. THEOR.

F a ſolid parallelepiped be cut by a plane parallel to two of its oppoſite planes; it divides the whole to two ſolids, the baſe of one of which ſhall be to e baſe of the other, as the one ſolid is to the other. See N.

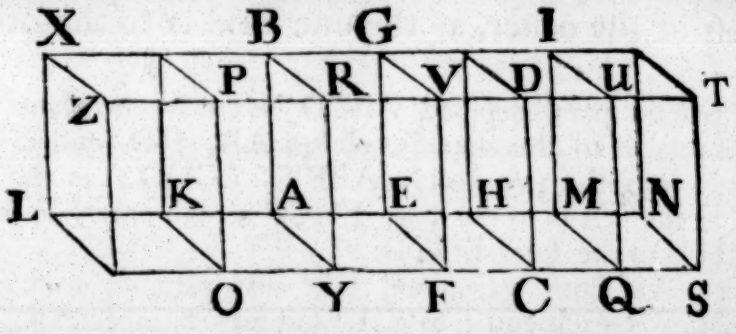
Let the ſolid parallelepiped ABCD be cut by the plane EV, ich is parallel to the oppoſite planes AR, HD, and divides whole into the two ſolids ABFV, EGCD; as the baſe EFY of the firſt is to the baſe EHCF of the other, ſo is the id ABFV to the ſolid EGCD.

Produce AH both ways, and take any number of ſtraight es HM, MN each equal to EH, and any number AK, KL ch equal to EA, and complete the parallelograms LO, KY, Q, MS, and the ſolids LP, KR, HU, MT: Then, becauſe ſtraight lines LK, KA, AE are all equal, the parallelograms



O, KY, AF are equal [a]: And likewiſe the parallelograms KX, [a] 36. 1.
B, AG [a]; as alſo [b] the parallelograms LZ, KP, AR, becauſe [b] 24. 11.
ey are oppoſite planes: For the ſame reaſon, the paral-
lograms EC, HQ, MS are equal [a]; and the parallelograms
G, HI, IN, as alſo [b] HD, MU, NT: Therefore three planes
the ſolid LP, are equal and ſimilar to three planes of the ſo-
KR, as alſo to three planes of the ſolid AV: But the three
anes oppoſite to theſe three are equal and ſimilar [b] to them
the ſeveral ſolids, and none of their ſolid angles are contain-
by more than three plane angles: Therefore the three ſolids
, KR, AV are equal [c] to one another: For the ſame reaſon, [c] C. 11.
three ſolids ED, HU, MT are equal to one another: There-
fore

Book XI. fore what multiple ſoever the baſe LF is of the baſe AF, t
ſame multiple is the ſolid LV of the ſolid AV: For the ſa
reaſon, whatever multiple the baſe NF is of the baſe HF, t
ſame multiple is the ſolid NV of the ſolid ED: And if the ba
[c] C. 11. LF be equal to the baſe NF, the ſolid LV is equal [c] to the ſ
lid NV; and if the baſe LF be greater than the baſe NF, t
ſolid LV is greater than the ſolid NV; and if leſs, leſs: Si
then there are four magnitudes, viz. the two baſes AF, F



and the two ſolids AV, ED, and of the baſe AF and ſolid A
the baſe LF and ſolid LV are any equimultiples whatever; a
of the baſe FH and ſolid ED, the baſe FN and ſolid NV are
ny equimultiples whatever; and it has been proved, that
the baſe LF is greater than the baſe FN, the ſolid LV is gre
than the ſolid NV; and if equal, equal; and if leſs, le
[d] 5. def. 5. Therefore [d] as the baſe AF is to the baſe FH, ſo is the ſolid A
to the ſolid ED. Wherefore, if a ſolid, &c. Q. E. D.

PROP. XXVI. PROB.

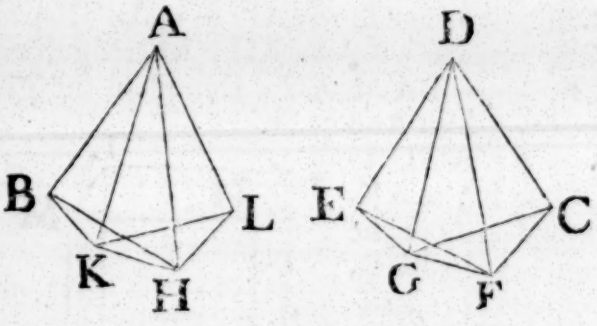
See N. AT a given point in a given ſtraight line, to ma
a ſolid angle equal to a given ſolid angle co
tained by three plane angles.

Let AB be a given ſtraight line, A a given point in it, a
D a given ſolid angle contained by the three plane angles ED
EDF, FDC: It is required to make at the point A in the ſtrai
line AB a ſolid angle equal to the ſolid angle D.

In the ſtraight line DF take any point F, from which d
[a] 11. 11. [a] FG perpendicular to the plane EDC, meeting that plane
G; join DG, and at the point A in the ſtraight line
[b] 23. 1. make [b] the angle BAL equal to the angle EDC, and in
plane BAL make the angle BAK equal to the angle ED
[c] 12. 11. then make AK equal to DG, and from the point K erect [c]

right angles to the plane BAL; and make KH equal to F, and join AH: Then the ſolid angle at A which is contain- by the three plane angles BAL, BAH, HAL is equal to the id angle at D contained by the three plane angles EDC, EDF, DC. Book XI.

Take the equal ſtraight lines AB, DE, and join HB, KB, , GE: And becauſe FG is perpendicular to the plane EDC, makes right angles [d] with every ſtraight line meeting it in at plane: Therefore each of the angles FGD, FGE is a right gle: For the ſame reaſon, HKA, HKB are right angles: And cauſe KA, AB are equal to GD, DE, each to each, and ntain equal angles, therefore the baſe BK is equal [e] to the ſe EG: And KH is equal to GF, and HKB, FGE, are right gles, therefore HB is equal [e] to FE: Again, becauſe AK, H are equal to DG, GF, and contain right angles, the ſe AH is equal to the baſe DF; and AB is equal to DE; erefore HA, AB are equal to FD, DE, and the baſe HB is ual to the baſe FE; erefore the angle AH is equal [f] to e angle EDF: For e ſame reaſon, the gle HAL is equal the angle FDC. cauſe if AL and C be made equal, d KL, HL, GC, C be joined, ſince the whole angle BAL is equal to the hole EDC, and the parts of them BAK, EDG are, by e conſtruction, equal; therefore the remaining angle KAL equal to the remaining angle GDC: And becauſe KA, L are equal to GD, DC, and contain equal angles, the baſe is equal [e] to the baſe GC: And KH is equal to GF, ſo that K, KH are equal to CG, GF, and they contain right angles; erefore the baſe HL is equal to the baſe FC: Again, becauſe A, AL are equal to FD, DC, and the baſe HL to the baſe C, the angle HAL is equal [f] to the angle FDC: Therefore, cauſe the three plane-angles BAL, BAH, HAL, which contain e ſolid angle at A, are equal to the three plane-angles EDC, DF, FDC, which contain the ſolid angle at D, each to each, d are ſituated in the ſame order; the ſolid angle at A is e- al [g] to the ſolid angle at D. Therefore at a given point in



[f] 8. 1.

[g] B. 11.

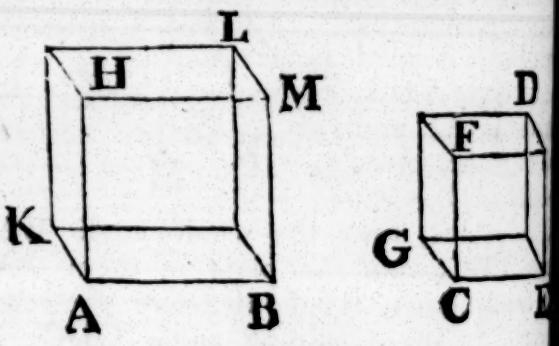
Book. XI. a given ſtraight line a ſolid angle has been made equal to a given ſolid angle contained by three plane angles. Which was to be done.

PROP. XXVII. PROB.

TO deſcribe from a given ſtraight line a ſolid parallelepiped ſimilar, and ſimilarly ſituated to one given.

Let AB be the given ſtraight line, and CD the given ſolid parallelepiped. It is required from AB to deſcribe a ſolid parallelepiped ſimilar, and ſimilarly ſituated to CD.

[a] 26. 11. At the point A of the given ſtraight line AB, make [a] a ſolid angle equal to the ſolid angle at C, and let BAK, KAH, HAB be the three plane angles which contain it, ſo that BAK be equal to the angle ECG, and KAH to GCF, and HAB to FCE: And as EC to CG, ſo make [b] BA to AK; and as GC to [b] 12. 6. CF, ſo make [b] KA to AH; wherefore, ex aequali [c], as EC to [c] 22. 5. CF, ſo is BA to AH: Complete the parallelogram BH, and the ſolid AL: And becauſe, as EC to CG, ſo BA to AK, the ſides about the equal angles ECG, BAK are proportionals; therefore the parallelogram BK is ſimilar to EG. For the ſame reaſon the parallelogram KH is ſimilar to GF, and HB to FE. Wherefore the three parallelograms of the ſolid AL are ſimilar to three of the ſolid CD; and the three oppoſite ones in each ſolid are equal [d] and ſimilar to theſe, each to each. Alſo, becauſe the plane angles which contain the ſolid angles of the figures are equal, each to each, and ſituated in the ſame order, the ſolid angles are equal [e], each to each. Therefore the ſolid AL is ſimilar [f] to the ſolid CD. Wherefore from a given ſtraight line AB a ſolid parallelepiped AL has been deſcribed ſimilar, and ſimilarly ſituated to the given one CD. Which was to be done.



[d] 24. 11.

[e] B. 11.

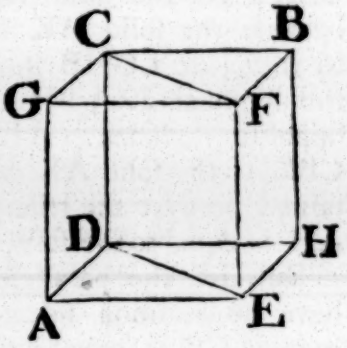
[f] 11. def. 11.

PRO

PROP. XXVIII. THEOR.

IF a ſolid parallelepiped be cut by a plane paſſing through the diagonals of two of the oppoſite planes; it ſhall be cut into two equal parts. See N.

Let AB be a ſolid parallelepiped, and DE, CF, the diagonals of the oppoſite parallelograms AH, GB, viz thoſe which are drawn betwixt the equal angles in each: And becauſe CD, GE are each of them parallel to GA, and not in the ſame plane with it, CD, EF are parallel [a]; wherefore the diagonals CF, DE are in the plane in which the parallels are, and are themſelves parallels [b]: And the plane CDEF ſhall cut the ſolid AB into two equal parts. [a] 9. 11.



[b] 16. 11.

Becauſe the triangle CGF is equal to the triangle CBF, and the triangle DAE to DHE; and that the parallelogram CA is equal [d] and ſimilar to the oppoſite one GE; and the parallelogram GE to CH: Therefore the priſm contained by the two triangles CGF, DAE, and the three parallelograms CA, GE, EC, is equal [e] to the priſm contained by the two triangles CBF, DHE, and the three parallelograms BE, CH, EC; becauſe they are contained by the ſame number of equal and ſimilar planes, alike ſituated, and none of their ſolid angles are contained by more than three plane angles. Therefore the ſolid AB is cut into two equal parts by the plane CDEF. Q. E. D. [c] 34. 1. [d] 24. 11. [e] C. 11.

'N B. The inſiſting ſtraight lines of a parallelepiped, mentioned in the next and ſome following propoſitions, are the ſides of the parallelograms betwixt the baſe and the oppoſite plane parallel to it.'

PROP. XXIX. THEOR.

SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines of which are terminated in the ſame ſtraight lines in the plane oppoſite to the baſe, are equal to one another. See N.

Let

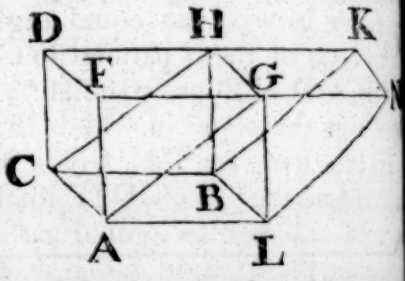
Book XI.

See the figures below.

Let the ſolid parallelepipeds AH, AK be upon the ſame base AB, and of the ſame altitude, and let their inſiſting ſtraight lines AF, AG, LM, LN, be terminated in the ſame ſtraight line FN, and CD, CE, BH, BK be terminated in the ſame ſtraight line DK; the ſolid AH is equal to the ſolid AK.

Firſt, Let the parallelograms DG, HN, which are oppoſite to the baſe AB, have a common ſide HG: Then, becauſe the ſolid AH is cut by the plane AGHC paſſing through the diagonals AG, CH of the oppoſite planes ALGF, CBHD, AH

[a] 28. 11. cut into two equal parts [a] by the plane AGHC: Therefore the ſolid AH is double of the priſm which is contained betwixt the triangles ALG, CBH: For the ſame reaſon, becauſe the ſolid AK is cut by the plane LGHB through the diagonals LG, BH of the oppoſite planes ALNG, CBKH, the ſolid AK is double of the ſame priſm which is contained betwixt the triangles ALG, CBH. Therefore the ſolid AH is equal to the ſolid AK.



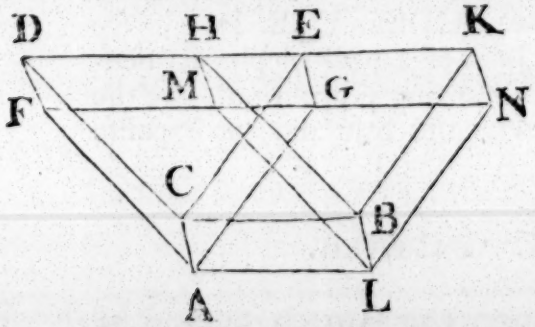
But, let the parallelograms DM, EN oppoſite to the baſe have no common ſide: Then, becauſe CH, CK are parallelograms, CB is equal [b] to each of the oppoſite ſides DH, EK wherefore DH is equal to EK: Add, or take away the common part HE; then DE is equal to HK: Wherefore alſo the triangle CDE is equal [c] to the triangle BHK: And the parallelogram DG is equal [d] to the parallelogram HN: For the ſame reaſon, the triangle AFG is equal to the triangle LMN, and the parallelogram CF is equal [e] to the parallelogram BM, and

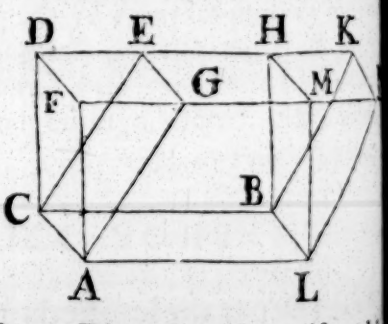
[b] 34. 1.

[c] 38. 1.

[d] 36. 1.

[e] 24. 11.





CG to BN; for they are oppoſite. Therefore the priſm which is contained by the two triangles AFG, CDE, and the three parallelograms AD, DG, GC is equal [f] to the priſm, contained by the two triangles LMN, BHK, and the three parrallelograms BM, MK, KL. If therefore the priſm LMNBHK

[f] C. 11.

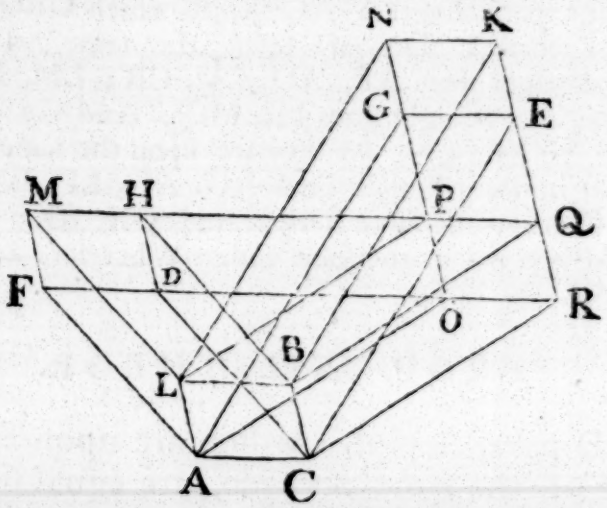
ken from the ſolid of which the baſe is the parallelogram B, and in which FDKN is the one oppoſite to it; and if om this ſame ſolid there be taken the priſm AFGCDE; e remaining ſolid, viz. the parallelepiped AH, is equal to the maining parallelepiped AK. Therefore ſolid parallelepipeds, c. Q. E. D. Book XI.

PROP. XXX. THEOR.

OLID parallelepipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines of hich are not terminated in the ſame ſtraight lines the plane oppoſite to the baſe, are equal to one nother. See N.

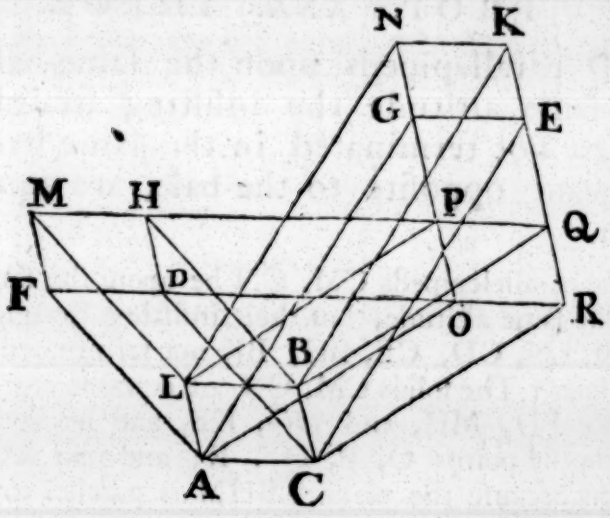
Let the parallelepipeds CM, CN be upon the ſame baſe AB, nd of the ſame altitude, but their inſiſting ſtraight lines AF, G, LM, LN, CD, CE, BH, BK not terminated in the ſame raight lines: The ſolids CM, CN are equal to one another.

Produce FD, MH, and NG, KE, and let them meet one nother in the points O, P, Q, R; and join AO, LP, BQ, R: And becauſe the plane LBHM is parallel to the oppoſite



lane ACDF, and that the plane LBHM is that in which are he parallels LB, MHPQ, in which alſo is the figure BLPQ; nd the plane ACDF is that in which are the parallels AC, OR, in which alſo is the figure CAOR; therefore the fi- ures BLPQ, CAOR are in parallel planes: In like manner, ecauſe the plane ALNG is parallel to the oppoſite plane CBKE, nd that the plane ALNG is that in which are the parallels AL,

Book XI. AL, OPGN, in which alſo is the figure ALPO; and the pl
CBKE is that in which are the parallels CB, RQEK, in wh
alſo is the figure CBQR; therefore the figures ALPO, CB
are in parallel planes: And the planes ACBL, ORQP are
rallel; therefore the ſolid CP is a parallelepiped: But the
lid CM of which the baſe is ACBL, to which FDHM is
[a] 29. 11. oppoſite parallelogram, is equal [a] to the ſolid CP of which



baſe is the parallelogram ACBL, to which ORQP is the
oppoſite; becauſe they are upon the ſame baſe, and
inſiſting ſtraight lines AF, AO, CD, CR; LM, LP, BH,
are in the ſame ſtraight lines FR, MQ: And the ſolid CP i
qual [a] to the ſolid CN, for they are upon the ſame baſe AC
and their inſiſting ſtraight lines AO, AG, LP, LN; CR,
BQ, BK are in the ſame ſtraight lines ON, RK: Therefore
ſolid CM is equal to the ſolid CN. Wherefore ſolid paralle
pipeds, &c. Q. E. D.

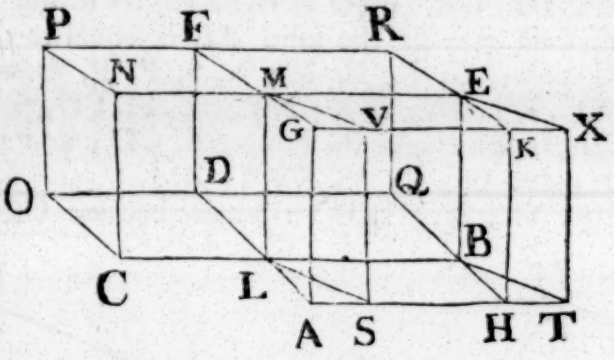
PROP. XXXI. THEOR.

See N. SOLID parallelepipeds which are upon equal baſ
and of the ſame altitude, are equal to one an
ther.

Let the ſolid parallelepipeds AE, CF, be upon equal baſes A
CD, and be of the ſame altitude; the ſolid AE is equal to
ſolid CF.

Firſt, Let the inſiſting ſtraight lines be at right angles to
baſes AB, CD, and let the baſes be placed in the ſame pla

d ſo as that the ſides CL, LB be in a ſtraight line; there- Book XI.
re the ſtraight line LM, which is at right angles to the plane
which the baſes are, in the point L, is common [a] to the two [a] 13. 11.
ids AE, CF; let the other inſiſting lines of the ſolids be
G, HK, BE; DF, OP, CN: And firſt, let the angle ALB be
ual to the angle CLD; then AL, LD are in a ſtraight line [b], [b] 14. 1.
oduce OD, HB, and let them meet in Q, and complete the
id parallelepiped LR the baſe of which is the parallelogram
Q, and of which LM is one of its inſiſting ſtraight lines:
herefore, becauſe the parallelogram AB is equal to CD, as the
ſe AB is to the baſe LQ, ſo is [c] the baſe CD to the ſame [c] 7. 5.
Q: And becauſe the ſolid parallelepiped AR is cut by the
ane LMEB, which is parallel to the oppoſite planes AK, DR;
the baſe AB is to the baſe LQ, ſo is [d] the ſolid AE to the [d] 25. 11.
lid LR: For the ſame reaſon, becauſe the ſolid parallelepiped
R is cut by the plane LMFD which is parallel to the oppoſite
anes CP, BR; as



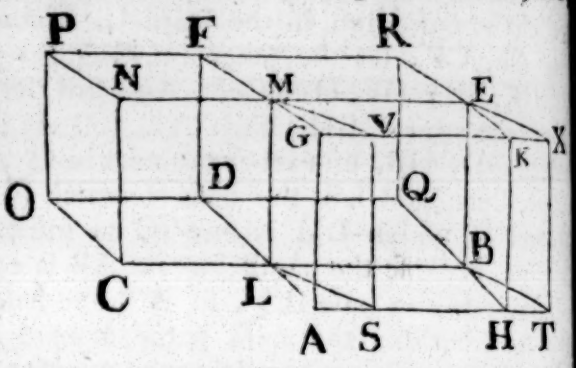
e baſe CD to the
aſe LQ, ſo is the
lid CF to the ſo-
d LR: But as the
aſe AB to the baſe
Q, ſo the baſe CD
the baſe LQ, as
efore was proved:
herefore as the
lid AE to the ſo-
d LR, ſo is the ſolid CF to the ſolid LR; and therefore the
lid AE is equal [e] to the ſolid CF. [e] 9. 5.

Bu tlet the ſolid parallelepipeds SE, CF be upon equal baſes
B, CD, and be of the ſame altitude, and let their inſiſting
raight lines be at right angles to the baſes; and place the
aſes SB, CD in the ſame plane, ſo that CL, LB be in a ſtraight
ne; and let the angles SLB, CLD be unequal; the ſolid SE
alſo in this caſe equal to the ſolid CF Produce DL, TS un-
l they meet in A, and from B draw BH parallel to DA; and
t HB, OD produced meet in Q, and complete the ſolids AE,
R: Therefore the ſolid AE of which the baſe is the parallelo-
ram LE, and AK the one oppoſite to it, is equal [f] to the ſo- [f] 29. 11.
id SE of which the baſe is LE, and to which SX is oppoſite;
r they are upon the ſame baſe LE, and of the ſame altitude,
nd their inſiſting ſtraight lines, viz LA, LO, BH, BT; MG,
MV, EK, EX are in the ſame ſtraight lines AT, GX: And be-

Book XI.

cauſe the parallelogram AB is equal [g] to SB, for they are u
on the ſame baſe LB, and between the ſame parallels LB, AT

[g] 35. 1.

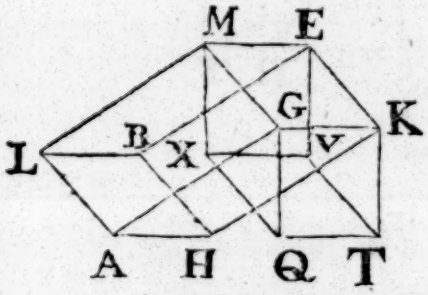


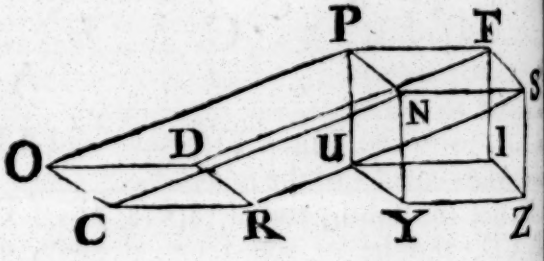
and that the baſe SB is equal to the baſe CD; therefore the baſe AB is equal to the baſe CD, and the angle ALB is equal to the angle CLD: Therefore, by the firſt caſe, the ſolid AE is equal to the ſolid CF; but the ſolid AE is equal to the ſolid SE, as was demon-ſtrated; therefore the ſolid SE is equal to the ſolid CF.

But if the inſiſting ſtraight lines AG, HK, BE, LM; CN RS, DF, CP, be not at right angles to the baſes AB, CD; i this caſe likewiſe the ſolid AE is equal to the ſolid CF: From the points G, K, E, M; N, S, F, P, draw the ſtraight line

[h] 11. 11.

GQ, KT, EV, MX; NY, SZ, FI, PU, perpendicular [h] to th plane in which are the baſes AB, CD; and let them meet it i the points Q, T, V, X; Y, Z, I, U, and join QT, TV, VX XQ; YZ, ZI, IU, UY: Then, becauſe GQ, KT, are at righ





[i] 6. 11.

angles to the ſame plane, they are parallel [i] to one another And MG, EK are parallels; therefore the planes MQ, ET, o which one paſſes through MG, GQ, and the other through EK, KT which are parallel to MG, GQ, and not in the ſame

[k] 15. 11.

plane with them, are parallel [k] to one another: For the ſame reaſon, the planes MV, GT are parallel to one another: There fore the ſolid QE is a parallelepiped: In like manner, it may be proved, that the ſolid YF is a parallelepiped: But, from what has been demonſtrated, the ſolid EQ is equal to the ſolid FY, becauſe they are upon equal baſes MK, PS, and of the ſame altitude, and have their inſiſting ſtraight lines at right angles

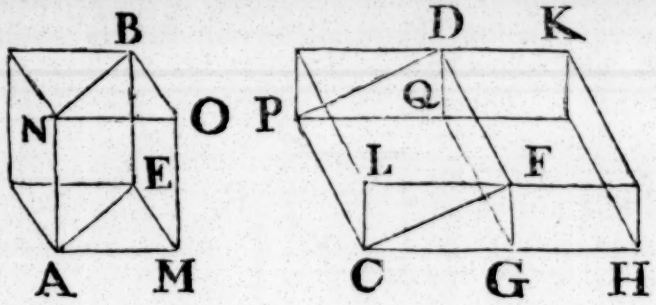
the bases: And the solid EQ is equal [1] to the solid AE; and Book XI.
e solid FY to the solid CF; because they are upon the same
ses and of the same altitude: Therefore the solid AE is equal [1] 29. or
the solid CF. Wherefore solid parallelepipeds, &c. Q. E. D. 30. 11.

PROP. XXXII. THEOR.

OLID parallelepipeds which have the same alti- See N.
tude, are to one another as their bases.

Let AB, CD be solid parallelepipeds of the same altitude: They are to one another as their bases; that is, as the base AE the base CF, so the solid AB to the solid CD.

To the straight line FG apply the parallelogram FH equal[a] [a] Cor. 45.
AE, so that the angle FGH be equal to the angle LCG; 1.
nd complete the solid parallelepiped GK upon the base FH, ne of whose insisting lines is FD, whereby the solids CD, GK ust be of the same altitude: Therefore the solid AB is equal [b] [b] 31. 11.
the solid GK, because they are upon equal bases AE, FH, and re of the same titude: And ecause the so- d parallelepi- ed CK is cut



y the plane DG which is parallel to its opposite planes, the base HF is [c] to the base FC, as the solid HD to the solid DC: But [c] 25. 11.
he base HF is equal to the base AE, and the solid GK to the olid AB: Therefore, as the base AE to the base CF, so is the olid AB to the solid CD. Wherefore solid parallelepipeds, &c. Q. E. D.

Cor. From this it is manifest that prisms upon triangular ba- es, of the same altitude, are to one another as their bases.

Let the prisms, the bases of which are the triangles AEM, FG, and NBO, PDQ the triangles opposite to them, have he same altitude; and complete the parallelograms AE, CF, nd the solid parallelepipeds AB, CD, in the first of which let MO, and in the other let GQ be one of the insisting lines. And ecause the solid parallelepipeds AB, CD have the same alti- itude, they are to one another as the base AE is to the base

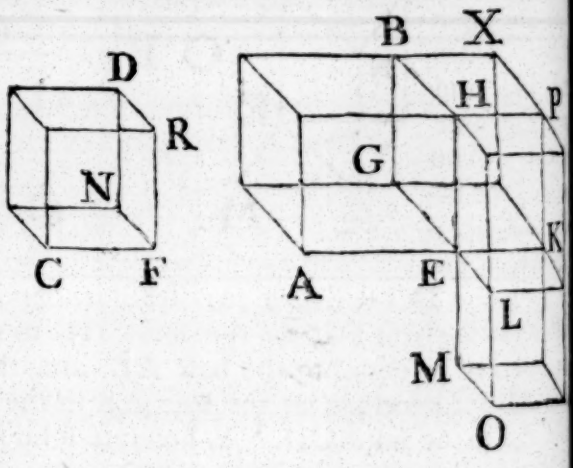
Book XI. CF; wherefore the prisms, which are their halves [d], are to one
another, as the base AE to the base CF; that is, as the triangle
[d] 28. 11. AEM to the triangle CFG.

PROP. XXXIII. THEOR.

SIMILAR solid parallelepipeds are one to another in the triplicate ratio of their homologous sides.

Let AB, CD be similar solid parallelepipeds, and the side AE homologous to the side CF: The solid AB has to the solid CD, the triplicate ratio of that which AE has to CF.

Produce AE, GE, HE, and in these produced take EK equal to CF, EL equal to FN, and EM equal to FR; and complete the parallelogram KL, and the solid KO: Because KE, EL are equal to CF, FN, and the angle KEL equal to the angle CFN because it is equal to the angle AEG which is equal to CFN, by reason that the solids AB, CD are similar; therefore the parallelogram KL is similar and equal to the parallelogram CN: For the same reason, the parallelogram MK is similar and equal to CR, and also OE to FD: Therefore three parallelograms of the solid KO are equal and similar to three parallelograms of the solid CD. And the three opposite ones in each solid
[a] 24. 11. are equal [a] and similar to these: Therefore the so-
[b] C. 11. lid KO is equal [b] and similar to the solid CD: Complete the parallelogram GK, and complete the solids EX, LP upon the bases GK, KL, so that EH be an insisting straight line in each of them, whereby they must be of the same altitude with the solid AB: And because the solids AB, CD are similar, and by permutation, as AE is to CF, so is EG to FN, and so is EK to FR; and FC is equal to EK, and FN to EL, and FR to EM; therefore as AE to EK, so is EG to EL, and so is HE
[c] 1. 6. to EM: But as AE to EK, so [c] is the parallelogram AG to the parallelogram GK; and as GE to EL, so [c] is GK to KL



and as HE to EM, so [c] is PE to KM: Therefore as the parallelogram AG to the parallelogram GK, so is GK to KL, and PE to KM: But as AG to GK, so [d] is the solid AB to the solid EX; and as GK to KL, so [d] is the solid EX to the solid PL; and as PE to KM, so [d] is the solid PL to the solid KO: And therefore as the solid AB to the solid EX, so is EX to PL, and PL to KO: But if four magnitudes be continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second: Therefore the solid AB has to the solid KO, the triplicate ratio of that which AB has to EX: But as AB is to EX, so is the parallelogram AG to the parallelogram GK, and the straight line AE to the straight line EK. Wherefore the solid AB has to the solid KO, the triplicate ratio of that which AE has to EK. And the solid KO is equal to the solid CD, and the straight line EK is equal to the straight line CF. Therefore the solid AB has to the solid CD, the triplicate ratio of that which the side AE has to the homologous side CF, &c. Q. E. D.

Book XI.

[c] 1. 6.

[d] 25. 11.

Cor. From this it is manifest, that if four straight lines be continual proportionals, as the first is to the fourth, so is the solid parallelepiped described from the first to the similar solid similarly described from the second; because the first straight line has to the fourth, the triplicate ratio of that which it has to the second.

PROP. D. THEOR.

SOLID parallelepipeds contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio which is the same with the ratio compounded of the ratios of their sides.

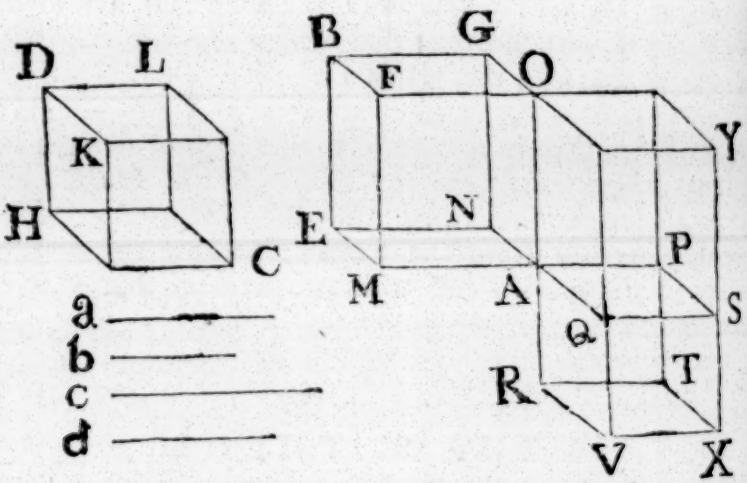
See N.

Let AB, CD be solid parallelepipeds, of which AB is contained by the parallelograms AE, AF, AG equiangular, each to each, to the parallelograms CH, CK, CL which contain the solid CD. The ratio which the solid AB has to the solid CD is the same with that which is compounded of the ratios of the sides AM to DL, AN to DK, and AO to DH.

Book XI. Produce MA, NA, OA to P, Q, R, ſo that AP be equal to DL, AQ to DK, and AR to DH; and complete the ſolid parallelepiped AX contained by the parallelograms AS, AT, AV ſimilar and equal to CH, CK, CL, each to each. Therefore the ſolid AX is equal [a] to the ſolid CD. Complete likewiſe the ſolid AY the baſe of which is AS, and of which AO is one of its inſiſting ſtraight lines. Take any ſtraight line a, and as MA to AP, ſo make a to b; and as NA to AQ, ſo make b to c; and as OA to AR, ſo c to d: Then, becauſe the parallelogram AE is equiangular to AS, AE is to AS, as the ſtraight line a to c, as is demonſtrated in the 23. Prop. Book 6 and the ſolids AB, AY, being betwixt the parallel planes BOY, EAS, are of the ſame altitude. Therefore the ſolid AB is to the ſolid AY, as [b] the baſe AE to the baſe AS; that is, as the ſtraight line a is to c. And the ſolid AY is to the ſolid

[a] C. 11.

[b] 32. 11.



AX, as [c] the baſe CQ is to the baſe QR; that is, as the ſtraight line OA to AR; that is, as the ſtraight line c to the ſtraight line d. And becauſe the ſolid AB is to the ſolid AY, as a is to c, and the ſolid AY to the ſolid AX, as c is to d; ex æquali the ſolid AB is to the ſolid AX, or CD which is equal to it, as the ſtraight line a is to d. But the ratio of a to d is ſaid to be compounded [d] of the ratios of a to b, b to c, and c to d, which are the ſame with the ratios of the ſides MA to AP, NA to AQ, and OA to AR, each to each. And the ſides AP, AQ, AR are equal to the ſides DL, DK, DH, each to each. Therefore the ſolid AB has to the ſolid CD the ratio which is the ſame with that which is compounded of the ratios of the ſides AM to DL, AN to DK, and AO to DH. Q. E. D.

[c] 25. 11.

[d] def. A. 5.

PROP

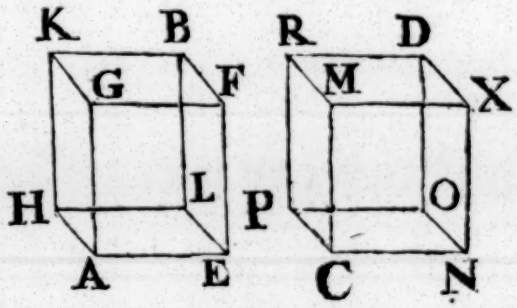
PROP. XXXIV. THEOR.

See N.

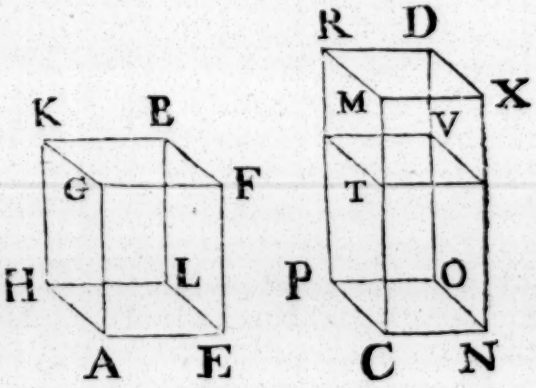
THE baſes and altitudes of equal ſolid parallelepipeds, are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the ſolid parallelepipeds are equal.

Let AB, CD be equal ſolid parallelepipeds; their baſes are reciprocally proportional to their altitudes; that is, as the baſe EH is to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB.

Firſt, Let the inſiſting ſtraight lines AG, EF, LB, HK; CM, NX, OD, PR be at right angles to the baſes. As the baſe EH to the baſe NP, ſo is CM to AG. If the baſe EH be equal to the baſe NP, then becauſe the ſolid AB is likewiſe equal to the ſolid CD, CM ſhall be equal to AG. Becauſe, if the baſes EH, NP be equal, but the altitudes AG, CM be not equal, neither ſhall the ſolid AB be equal to the ſolid CD. But the ſolids are equal, by the hypotheſis. Therefore the altitude CM is not unequal to the altitude AG; that is, they are equal. Wherefore as the baſe EH to the baſe NP, ſo is CM to AG.



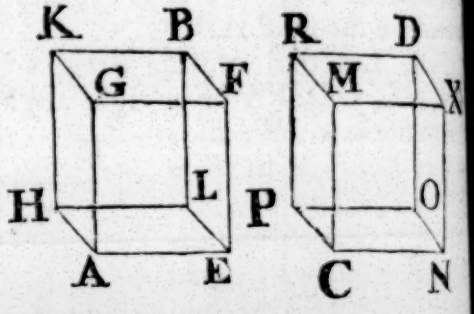
Next, Let the baſes EH, NP not be equal, but EH greater than the other: Since then the ſolid AB is equal to the ſolid CD, CM is therefore greater than AG: For if it be not, neither alſo, in this caſe, would the ſolids AB, CD be equal, which, by the hypotheſis, are equal. Make then CT equal to AG, and complete the ſolid parallelepiped CV of which the baſe is NP, and altitude CT. Becauſe the ſolid AB is equal to the ſolid CD, therefore the ſolid AB is to the



Book XI. solid CV, as [a] the solid CD to the solid CV. But as the solid AB to the solid CV, so [b] is the base EH to the base NP; for the solids AB, CV are of the same altitude; and as the solid CD, to CV, so [c] is the base MP to the base PT, and so [d] the straight line MC to CT; and CT is equal to AG. Therefore, as the base EH to the base NP, so is MC to AG. Wherefore the bases of the solid parallelepipeds AB, CD are reciprocally proportional to their altitudes.

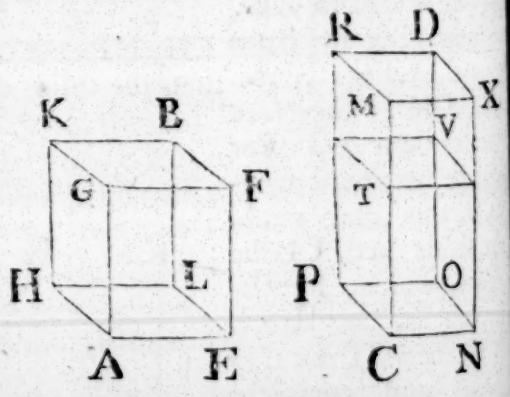
[a] 7. 5.
[b] 32. 11.
[c] 25. 11.
[d] 1. 6.

Let now the bases of the solid parallelepipeds AB, CD be reciprocally proportional to their altitudes; viz. as the base EH to the base NP, so the altitude of the solid CD to the altitude of the solid AB; the solid AB is equal to the solid CD. Let the insisting lines be, as before, at right angles to the bases. Then, if the base EH be equal to the base NP, since EH is to NP, as the altitude of the solid CD is to the altitude of the solid AB, therefore the altitude of CD is equal [e] to the altitude of AB. But solid parallelepipeds upon equal bases, and of the same altitude, are equal [f] to one another; therefore the solid AB is equal to the solid CD.

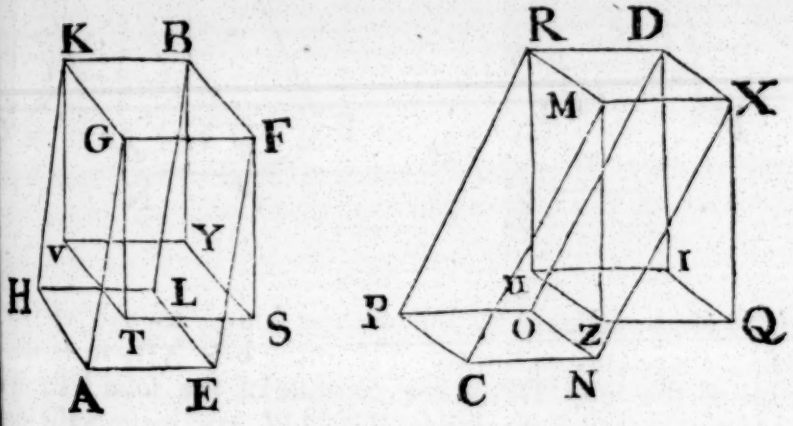


[e] A. 5.
[f] 31. 11.

But let the bases EH, NP be unequal, and let EH be the greater of the two. Therefore, since as the base EH to the base NP, so is CM the altitude of the solid CD to AG the altitude of AB, CM is greater [e] than AG. Again, Take CT equal to AG, and complete, as before, the solid CV. And, because the base EH is to the base NP, as CM to AG, and that AG is equal to CT, therefore the base EH is to the base NP, as MC to CT. But as the base EH is to NP, so [b] is the solid AB to the solid CV; for the solids AB, CV are of the same altitude; and as MC to CT, so is the base MP to the base PT,



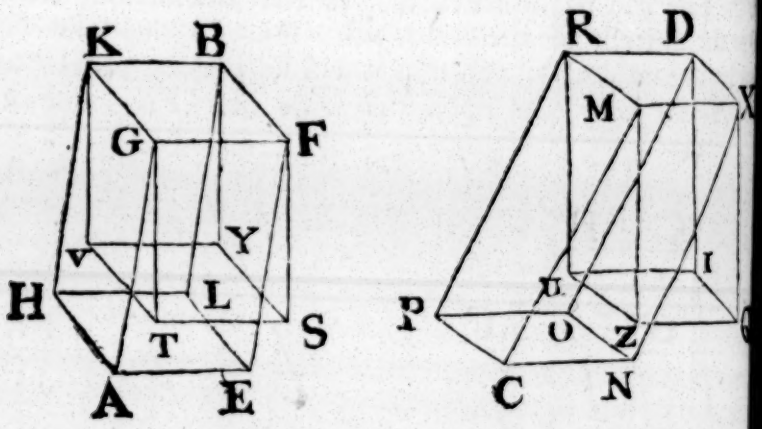
, and the ſolid CD to the ſolid [c] CV: And therefore, as the Book XI.
d AB to the ſolid CV, ſo is the ſolid CD to the ſolid CV;
t is, each of the ſolids AB, CD has the ſame ratio to the [c] 25. 11.
d CV; and therefore the ſolid AB is equal to the ſolid CD.
Second general caſe. Let the inſiſting ſtraight lines FE,
, GA, KH; XN, DO, MC, RP not be at right angles to
baſes of the ſolids; and from the points F, B, K, G; X,
R, M draw perpendiculars to the planes in which are the
es EH, NP, meeting thoſe planes in the points S, Y, V, T;
I, U, Z; and complete the ſolids FV, XU, which are pa-
elepipeds, as was proved in the laſt part of prop. 31ſt of
s book. In this caſe likewiſe, if the ſolids AB, CD be e-
l, their baſes are reciprocally proportional to their altitudes,
. the baſe EH to the baſe NP, as the altitude of the ſolid
) to the altitude of the ſolid AB. Becauſe the ſolid AB is
al to the ſolid CD, and that the ſolid BT is equal [f] to the [f] 29. or
d BA, for they are upon the ſame baſe FK, and of the 30. 11.



e altitude; and that the ſolid DC is equal [g] to the ſolid
, being upon the ſame baſe XR, and of the ſame altitude;
refore the ſolid BT is equal to the ſolid DZ: But the baſes
reciprocally proportional to the altitudes of equal ſolid pa-
elepipeds of which the inſiſting ſtraight lines are at right
les to their baſes, as before was proved: Therefore, as the
FK to the baſe XR, ſo is the altitude of the ſolid DZ to
altitude of the ſolid BT: And the baſe FK is equal to the
EH, and the baſe XR to the baſe NP: Wherefore, as the
EH to the baſe NP, ſo is the altitude of the ſolid DZ to
altitude of the ſolid BT: But the altitudes of the ſolids
, DC, as alſo of the ſolids BT, BA are the ſame. There-
, as the baſe EH to the baſe NP, ſo is the altitude of the
ſolid

Book XI. ſolid CD to the altitude of the ſolid AB; that is, the baſes the ſolid parallelepipeds AB, CD are reciprocally proportio to their altitudes.

Next, Let the baſes of the ſolids AB, CD be reciproc proportional to their altitudes, viz. the baſe EH to the NP, as the altitude of the ſolid CD to the altitude of the lid AB; the ſolid AB is equal to the ſolid CD: The f conſtruction being made; becauſe, as the baſe EH to the NP, ſo is the altitude of the ſolid CD to the altitude of ſolid AB; and that the baſe EH is equal to the baſe FK; NP to XR; therefore the baſe FK is to the baſe XR, a altitude of the ſolid CD to the altitude of AB: But the



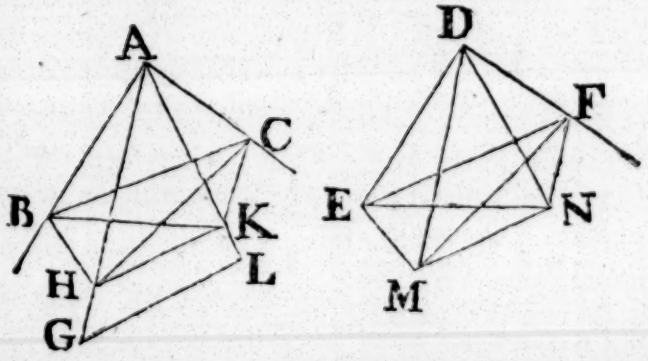
tudes of the ſolids AB, BT are the ſame, as alſo of CD DZ; therefore, as the baſe FK to the baſe XR, ſo is the tude of the ſolid DZ to the altitude of the ſolid BT: W fore the baſes of the ſolids BT, DZ are reciprocally pro tional to their altitudes; and their inſiſting ſtraight lines a right angles to the baſes; wherefore, as was before proved, [s] 29. or 30. 11. ſolid BT is equal to the ſolid DZ: But BT is equal [s] to th lid BA, and DZ to the ſolid DC, becauſe they are upon ſame baſes, and of the ſame altitude. Therefore the ſolid is equal to the ſolid CD. Q. E. D

PROP. XXXV. THEOR.

F, from the vertices of two equal plane angles there be drawn two ſtraight lines elevated above e planes in which the angles are, and containing ual angles with the ſides of thoſe angles, each to ch; and if in the lines above the planes there be ken any points, and from them perpendiculars be awn to the planes in which the firſt named angles e: And from the points in which they meet the nes, ſtraight lines be drawn to the vertices of the gles firſt named; theſe ſtraight lines ſhall contain ual angles with the ſtraight lines which are above e planes of the angles.

See N.

Let BAC, EDF be two equal plane angles; and from the nts A, D let the ſtraight lines AG, DM be elevated above planes of the angles, making equal angles with their ſides, h to each, viz the angle GAB equal to the angle MDE, and AC to MDF; and in AG, DM let any points G, M be ta-, and from them let perpendiculars GL, MN be drawn he planes BAC, EDF meeting theſe planes in the points



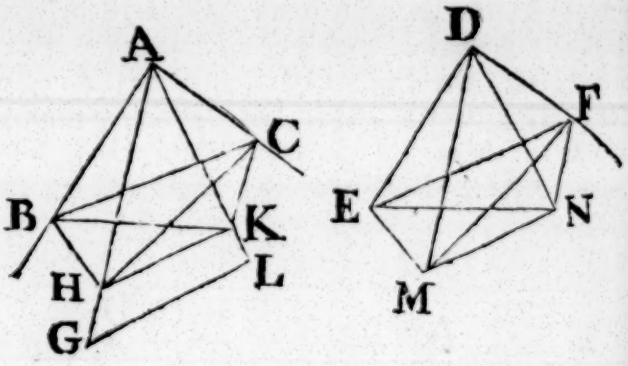
N; and join LA, ND: The angle GAL is equal to the le MDN.

Make AH equal to DM, and through H draw HK parallel GL: But GL is perpendicular to the plane BAC; where- HK is perpendicular [a] to the ſame plane: From the points N, to the ſtraight lines AB, AC, DE, DF, draw perpen- lars KB, KC, NE, NF; and join HB, BC, ME, EF:

[a] 8. 11.

Becauſe

Book XI. Becauſe HK is perpendicular to the plane BAC, the p
HBK which paſſes through HK is at right angles [b] to the p
[b] 18. 11. BAC: and AB is drawn in the plane BAC at right angl
the common ſection BK of the two planes; therefore A
[c] 4. def. 11. perpendicular [c] to the plane HBK, and makes right ang
with every ſtraight line meeting it in that plane: But BH
[d] 3. def. 11. it in that plane; therefore ABH is a right angle: For the
reaſon, DEM is a right angle, and is therefore equal to
angle ABH: And the angle HAB is equal to the angle M
Therefore in the two triangles HAB, MDE there are two a
in one equal to two angles in the other, each to each,
one ſide equal to one ſide, oppoſite to one of the equal a
in each, viz. HA equal to DM; therefore the remaining
[e] 26. 2. are equal [e], each to each: Wherefore AB is equal to DE.
the ſame manner, if HC and MF be joined, it may be de
ſtrated that AC is equal to DF. Therefore, ſince AB is equ
DE, BA and AC are equal to ED and DF; and the a



BAC is equal to the angle EDF; wherefore the baſe BC
[f] 4. 1. qual [f] to the baſe EF, and the remaining angles to the re
ing angles: The angle ABC is therefore equal to the
DEF: And the right angle ABK is equal to the right
DEN, whence the remaining angle CBK is equal to th
maining angle FEN: For the ſame reaſon, the angle BC
qual to the angle EFN: Therefore, in the two triangles
EFN, there are two angles in one equal to two angles
other, each to each, and one ſide equal to one ſide ad
to the equal angles in each, viz. BC equal to EF; the
ſides therefore are equal to the other ſides; BK then is
to EN: And AB is equal to DE; wherefore AB, BK are
to DE, EN; and they contain right angles; wherefo
baſe AK is equal to the baſe DN: And ſince AH is eq

, the ſquare of AH is equal to the ſquare of DM: But the Book XI.
ares of AK, KH are equal to the ſquare [g] of AH, becauſe
H is a right angle: And the ſquares of DN, NM are equal [g] 47. 1.
he ſquare of DM, for DNM is a right angle: Wherefore
ſquares of AK, KH are equal to the ſquares of DN, NM;
of thoſe the ſquare of AK is equal to the ſquare of DN:
erefore the remaining ſquare of KH is equal to the remain-
ſquare of NM; and the ſtraight line KH to the ſtraight
NM: And becauſe HA, AK are equal to MD, DN, each
ach, and the baſe HK to the baſe MN, as has been proved;
erefore the angle HAK is equal [h] to the angle MDN. Q. [h] 8. 1.
D.

Cor. From this it is manifeſt, that if, from the vertices of
equal plane angles, there be elevated two equal ſtraight lines
aining equal angles with the ſides of the angles, each to
; the perpendiculars drawn from the extremities of the e-
ſtraight lines to the planes of the firſt angles are equal to
another.

Another Demonſtration of the Corollary.

Let the plane angles BAC, EDF be equal to one another,
let AH, DM be two equal ſtraight lines above the planes
the angles, containing equal angles with BA, AC; ED,
, each to each, viz. the angle HAB equal to MDE, and
AC equal to the angle MDF; and from H, M let HK, MN
perpendiculars to the planes BAC, EDF: HK is equal to
N.

Becauſe the ſolid angle at A is contained by the three plane
les BAC, BAH, HAC, which are, each to each, equal
he three plane angles EDF, EDM, MDF containing the
d angle at D; the ſolid angles at A and D are equal: And
erefore coincide with one another; to wit, if the plane angle
C be applied to the plane angle EDF, the ſtraight line AH
ncides with DM, as was ſhewn in prop B. of this book:
d becauſe AH is equal to DM, the point H coincides with
point M: Wherefore HK which is perpendicular to the
ne BAC coincides with [i] MN which is perpendicular to the [i] 13. 11.
ne EDF, becauſe theſe planes coincide with one another:
erefore HK is equal to MN. Q. E. D.

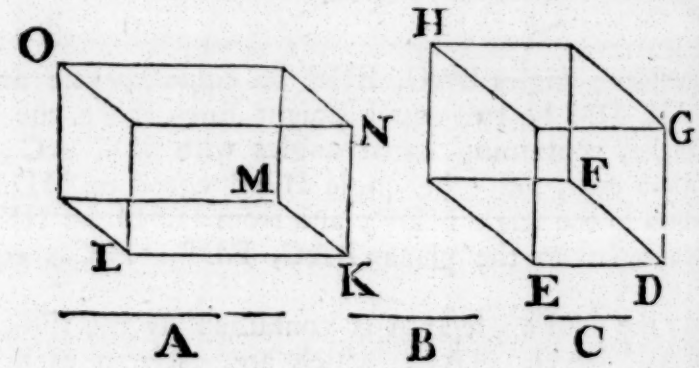
PROP.

PROP. XXXVI. THEOR.

See N. IF three ſtraight lines be proportionals, the ſo parallelepiped deſcribed from all three as its ſid is equal to the equilateral parallelepiped deſcrib from the mean proportional, one of the ſolid ang of which is contained by three plane angles equ each to each, to the three plane angles containi one of the ſolid angles of the other figure.

Let A, B, C be three proportionals, viz. A to B, as B C. The ſolid deſcribed from A, B, C is equal to the equ teral ſolid deſcribed from B, equiangular to the other.

Take a ſolid angle D contained by three plane angles E FDG, GDE; and make each of the ſtraight lines ED, D DG equal to B, and complete the ſolid parallelepiped D Make LK equal to A, and at the point K in the ſtraight LK make [a] a ſolid angle contained by the three plane an
[a] 26. 11. LKM, MKN, NKL equal to the angles EDF, FDG, GD



Each to each; and make KN equal to B, and KM equa C; and complete the ſolid parallelepiped KO: And becauſe A is to B, ſo is B to C, and that A is equal to LK, an to each of the ſtraight lines DE, DF, and C to KM; th fore LK is to ED, as DF to KM; that is, the ſides about equal angles are reciprocally proportional; therefore the
[b] 14. 6. rallelogram LM is equal [b] to EF: And becauſe EDF, LKM two equal plane angles, and the two equal ſtraight lines KN are drawn from their vertices above their planes, and tain equal angles with their ſides; therefore the perpend lars from the points G, N, to the planes EDF, LKM a

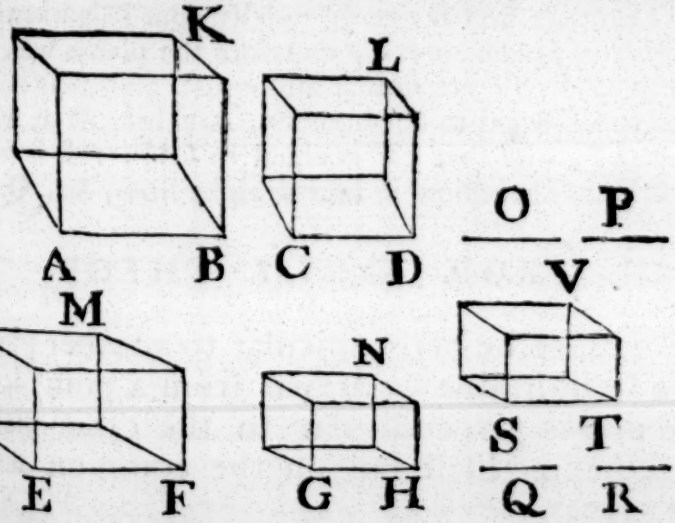
[c] to one another: Therefore the ſolids KO, DH are of Book XI.
ſame altitude; and they are upon equal baſes LM, EF,
therefore they are equal [d] to one another: But the ſolid [c] Cor. 35.
is deſcribed from the three ſtraight lines A, B, C, and the 11.
DH from the ſtraight line B. If therefore three ſtraight [d] 31. 11.
s, &c. Q. E. D.

PROP. XXXVII. THEOR.

four ſtraight lines be proportionals, the ſimilar See N.
ſolid parallelepipeds ſimilarly deſcribed from them
ll alſo be proportionals. And if the ſimilar paral-
pipeds ſimilarly deſcribed from four ſtraight lines
proportionals, the ſtraight lines ſhall be propor-
nals.

et the four ſtraight lines AB, CD, EF, GH be proporti-
s, viz. as AB to CD, ſo EF to GH; and let the ſimilar
llelepipeds AK, CL, EM, GN be ſimilarly deſcribed from
n. AK is to CL, as EM to GN.

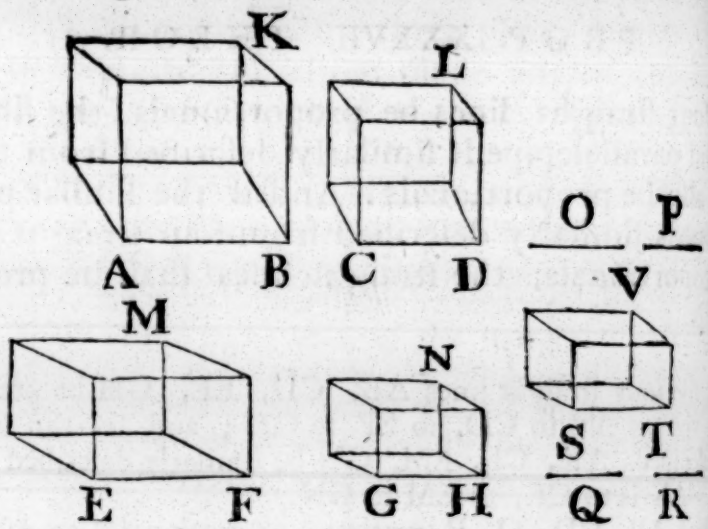
Take [a] AB, CD, O, P continual proportionals, as alſo EF, [a] 11. 6.
, Q, R: And becauſe as AB is to CD, ſo EF to GH;



is [b] to O, as GH to Q, and O to P, as Q to R; there- [b] 11. 5.
ex aequali [c], AB is to P, as EF to R: But as AB to P, [c] 22 5.
s the ſolid AK to the ſolid CL; and as EF to R, ſo [d] is [d] Cor 33.
olid EM to the ſolid GN: Therefore [b] as the ſolid AK to 11.
olid CL, ſo is the ſolid EM to the ſolid GN.

But

Book XI. But let the ſolid AK be to the ſolid CL, as the ſolid EM
the ſolid GN: The ſtraight line AB is to CD, as EF to GH
[e] 27. 11. Take AB to CD, as EF to ST, and from ST deſcribe
ſolid parallelepiped SV ſimilar and ſimilarly ſituated to either
the ſolids EM, GN: And becauſe AB is to CD, as EF to S
and that from AB, CD the ſolid parallelepipeds AK, CL
ſimilarly deſcribed; and in like manner the ſolids EM,
from the ſtraight lines EF, ST; therefore AK is to CL,



EM to SV: But, by the hypotheſis, AK is to CL, as EM
[f] 9. 5. GN: Therefore GN is equal [f] to SV: But it is likewiſe ſim
and ſimilarly ſituated to SV; therefore the planes which con
the ſolids GN, SV are ſimilar and equal, and their homolog
ſides GH, ST equal to one another: And becauſe as AB to C
ſo EF to ST, and that ST is equal to GH; AB is to CD,
EF to GH. Therefore, if four ſtraight lines, &c. Q. E. D

PROP. XXXVIII. THEOR.

See N. "IF a plane be perpendicular to another plane, a
" a ſtraight line be drawn from a point in one
" the planes perpendicular to the other plane, t
" ſtraight line ſhall fall on the common ſection
" the planes.

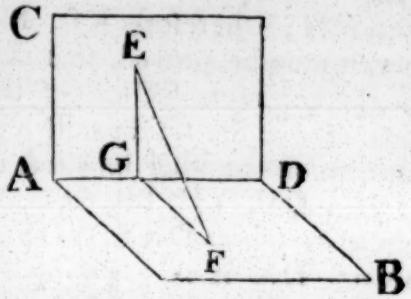
" Let the plane CD be perpendicular to the plane AB,
" let AD be their common ſection; if any point E be t
" in the plane CD, the perpendicular drawn from E to
" plane AB ſhall fall on AD.

" For, if it does not, let it, if poſſible, fall elſewhere, as EF; and let it meet the plane AB in the point F; and from F draw [a], in the plane AB, a perpendicular FG to DA, which is alſo perpendicular [b] to the plane CD; and join EG: Then becauſe FG is perpendicular to the plane CD, and the ſtraight line EG, which is in that plane, meets it; therefore FGE is a right angle [c]: But EF is alſo at right angles to the plane AB; and therefore EFG is a right angle: Wherefore two of the angles of the triangle EFG are equal together to two right angles; which is abſurd: Therefore the perpendicular from the point E to the plane AB does not fall elſewhere than upon the ſtraight line AD: It therefore falls upon it. If therefore a plane, &c. Q. E. D."

Book XI.

[a] 12. 1.

[b] 4. def. 11.



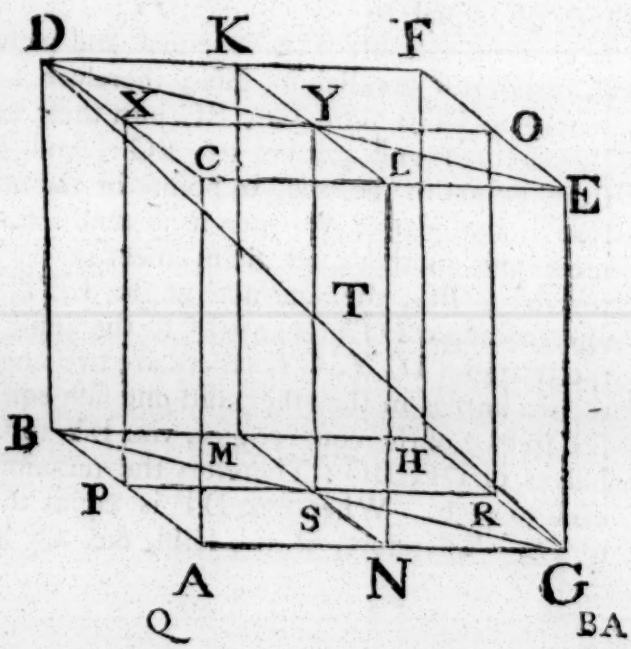
[c] 3. def. 11.

PROP. XXXIX. THEOR.

IN a ſolid parallelepiped, if the ſides of two of the oppoſite planes be divided each into two equal parts, the common ſection of the planes paſſing through the points of diviſion, and the diameter of the ſolid parallelepiped cut each other into two equal parts.

See N.

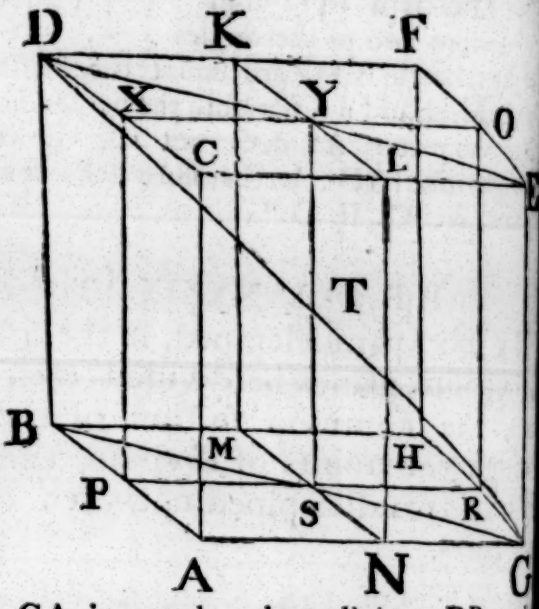
Let the ſides of the oppoſite planes EF, AH of the ſolid parallelepiped CF, be divided each into two equal parts in the points K, L, M, N; X, O, P, R: And join KL, MN, XO, PR: And becauſe DK, CL are equal and parallel, KL is parallel [a] to DC: For the ſame reaſon, MN is parallel to BA: And



[a] 33. 1.

Book XI. BA is parallel to DC; therefore becauſe KL, BA are each of them parallel to DC, and not in the ſame plane with it, KL is
[b] 9. 11. parallel [b] to BA: And becauſe KL, MN are each of them parallel to BA, and not in the ſame plane with it, KL is parallel [b] to MN; wherefore KL, MN are in one plane. In like manner, it may be proved, that XO, PR are in one plane. Let YS be the common ſection of the planes KN, XR; and DG the diameter of the ſolid parallelepiped AF. YS and DG do meet and cut one another into two equal parts.

Join DY, YE, BS, SG. Becauſe DX is parallel to OE, the
[c] 29. 1. alternate angles DXY, YOE are equal [c] to one another: And becauſe DX is equal to OE, and XY to YO, and contain equal angles, the baſe DY
[d] 4. 1. is equal [d] to the baſe YE, and the other angles are equal; therefore the angle XYD is equal to the angle OYE, and DYE is a ſtraight
[e] 14. 1 [e] line: For the ſame reaſon BSG is a ſtraight line, and BS equal to SG: And becauſe CA is equal and parallel to DB, and ſo equal and parallel to EG; therefore DB is equal and parallel [b] to EG: And DE, BG join their extremities; therefore



[a] 33. 1. DE is equal and parallel [a] to BG: And DG, YS are drawn from points in the one, to points in the other; and are therefore in one plane: Whence it is manifeſt, that DG, YS muſt meet one another; let them meet in T: And becauſe DE is parallel to BG, the alternate angles EDT, BGT are equal
[f] 15. 1. and the angle DTY is equal [f] to the angle GTS: Therefore in the triangles DTY, GTS there are two angles in the one equal to two angles in the other, and one ſide equal to one ſide, oppoſite to two of the equal angles, viz. DY to GS; for they are
[g] 26. 1. halves of DF, BG: Therefore the remaining ſides are equal each to each. Wherefore DT is equal to TG, and YT equal to TS. Therefore, if in a ſolid, &c. Q. E. D.

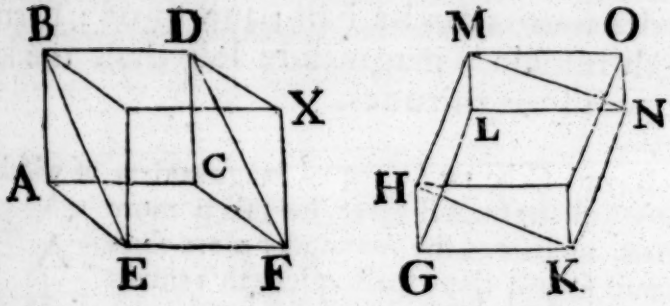
PRO

PROP. XL. THEOR.

' there be two triangular priſms of the ſame altitude, the baſe of one of which is a parallelom, and the baſe of the other a triangle; if the rallelogram be double of the triangle, the priſms ll be equal to one another.

et the priſms ABCDEF, GHKLMN be of the ſame altie, the firſt whereof is contained by the two triangles ABE, F, and the three parallelograms AD, DE, EC; and the o- by the two triangles GHK, LMN and the three parallems LH, HN, NG; and let one of them have a parallem AF, and the other a triangle GHK for its baſe; if the llelogram AF be double of the triangle GHK, the priſm CDEF is equal to the priſm GHKLMN.

Complete the ſolids AX, GO; and becauſe the parallelom AF is double of the triangle GHK; and the parallelo-



HK double [a] of the ſame triangle; therefore the paral- [a] 34. 1.
gram AF is equal to HK. But ſolid parallelepipeds upon l baſes, and of the ſame altitude, are equal [b] to one an- [b] 31. 11.
r. Therefore the ſolid AX is equal to the ſolid GO; and riſm ABCDEF is half [c] of the ſolid AX; and the priſm [c] 28. 11.
KLMN half [c] of the ſolid GO. Therefore the priſm CDEF is equal to the priſm GHKLMN. Wherefore, if be two, &c. Q. E. D.

THE ELEMENTS OF EUCLID

See N.

BOOK XII.

LEMMA I.

Which is the firſt Propoſition of the tenth Book, and is neceſſ
to ſome of the Propoſitions of this Book.

IF from the greater of two unequal magnitud
there be taken more than its half, and from t
remainder more than its half; and ſo on: There ſh
at length remain a magnitude leſs than the leaſt
the propoſed magnitudes.

Let AB and C be two unequal magnitudes, of which AB
the greater. If from AB there be taken more than its half, and from the remainder more than its half, and ſo on; there ſhall at length remain a magnitude leſs than C.

For C may be multiplied ſo, as at length to become greater than AB. Let it be ſo multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and ſo on, until there be as many diviſions in AB as there are in DE: And let the diviſions in AB be AK, KH, HB; and the diviſions in DE be DF, FG, GE. And becauſe DE is greater than AB, and

t EG taken from DE is not greater than its half, but BH taken Book XII.
m AB is greater than its half; therefore the remainder GD
greater than the remainder HA. Again, becauſe GD is
ater than HA, and that GF is not greater than the half of
D, but HK is greater than the half of HA; therefore the re-
inder FD is greater than the remainder AK. And FD is
ual to C, therefore C is greater than AK; that is, AK is leſs
an C. Q. E. D.

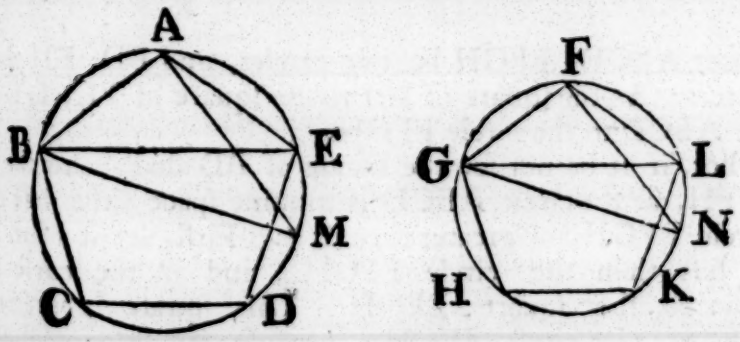
And if only the halves be taken away, the ſame thing may the ſame way be demonſtrated.

PROP. I. THEOR.

IMILAR polygons inſcribed in circles, are to one another as the ſquares of their diameters.

Let ABCDE, FGHKL be two circles, and in them the ſi-
ilar polygons ABCDE, FGHKL; and let BM, GN be the
meters of the circles: As the ſquare of BM is to the ſquare
GN, ſo is the polygon ABCDE to the polygon FGHKL.

Join BE, AM, GL, FN: And becauſe the polygon ABCDE is
ilar to the polygon FGHKL, and ſimilar polygons are divided
o ſimilar triangles; the triangles ABE, FGL are ſimilar and e-

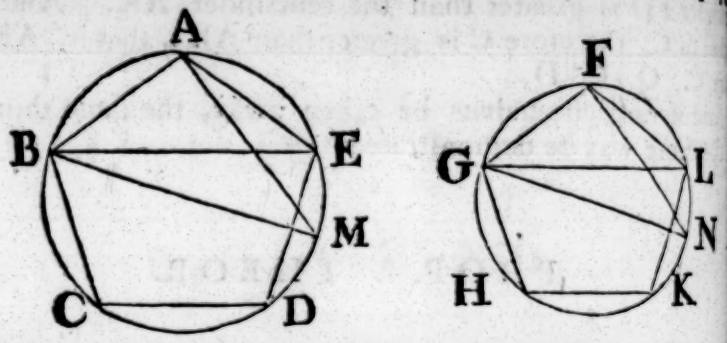


iangular [b]; and therefore the angle AEB is equal to the angle [b] 6. 7.
LG: But AEB is equal [c] to the AMB, becauſe they ſtand up- [c] 21. 3.
the ſame circumference; and the angle FLG is, for the ſame
ſon, equal to the angle FNG: Therefore alſo the angle AMB
equal to FNG: And the right angle BAM is equal to the
ght [d] angle GFN; wherefore the remaining angles in the tri- [d] 31. 3.
gles ABM, FGN are equal, and they are equiangular to one

Book XII.

another: Therefore as BM to GN, so [e] is BA to GF; and therefore the duplicate ratio of BM to GN, is the same [f] with the duplicate ratio of BA to GF: But the ratio of the square of BM to the square of GN, is the duplicate [g] ratio of that which BM has to GN; and the ratio of the polygon ABCDE to the polygon

[e] 4. 6.
[f] 10. def. 5. & 22. 5.
[g] 20. 6.



FGHKL is the duplicate [g] of that which BA has to GF: Therefore as the square of BM to the square of GN, so is the polygon ABCDE to the polygon FGHKL. Wherefore similar polygons &c. Q. E. D.

PROP. II. THEOR.

See N.

CIRCLES are to one another as the squares of their diameters.

Let ABCD, EFGH be two circles, and BD, FH their diameters: As the square of BD to the square of FH, so is the circle ABCD to the circle EFGH.

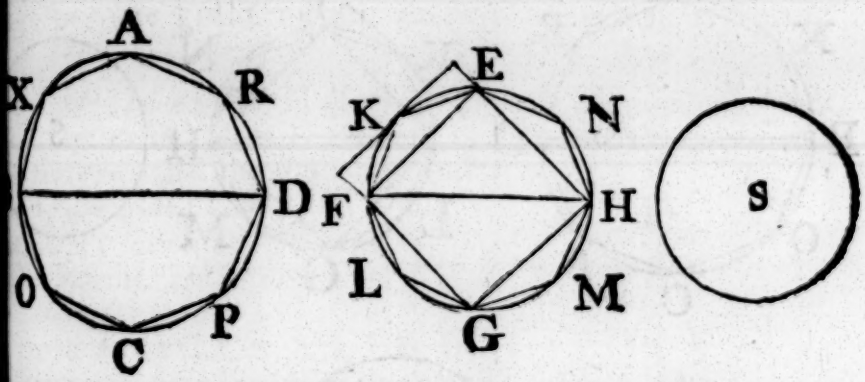
For, if it be not so, the square of BD shall be to the square of FH, as the circle ABCD is to some space either less than the circle EFGH, or greater than it *. First, let it be to a space S less than the circle EFGH; and in the circle EFGH describe the square EFGH: This square is greater than half of the circle EFGH; because if, through the points E, F, G, H, there be drawn tangents to the circle,

* For there is some square equal to the circle ABCD; let P be the side of it, and to three straight lines BD, FH and P, there can be a fourth proportional; let this be Q: Therefore the squares of these four straight lines are proportionals; that is, to the squares of BD, FH the circle ABCD, it is possible there be a fourth proportional. Let this And in like manner are to be understood some things in some of the following propositions.

uare EFGH is half [a] of the ſquare deſcribed about the circle; and the circle is leſs than the ſquare deſcribed about it; therefore the ſquare EFGH is greater than half of the circle. Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE: Therefore each of the triangles EKF, FLG, GMH, HNE is greater than half of the ſegment of the circle it ſtands in; becauſe if ſtraight lines touching the circle be drawn through the points K, L, M, N, and parallelograms upon the ſtraight lines EF, FG, GH, HE be completed; each of the triangles EKF, FLG, GMH, HNE ſhall be the half [a] of the parallelogram in which it is: But every ſegment is leſs than the parallelogram in which it is: Wherefore each of the triangles EKF, FLG, GMH, HNE is greater than half the ſegment of the circle which contains it: And if theſe circumferences before named be divided each into two equal parts, and their extremities be joined by ſtraight lines, by continuing

[a] 47. 1.

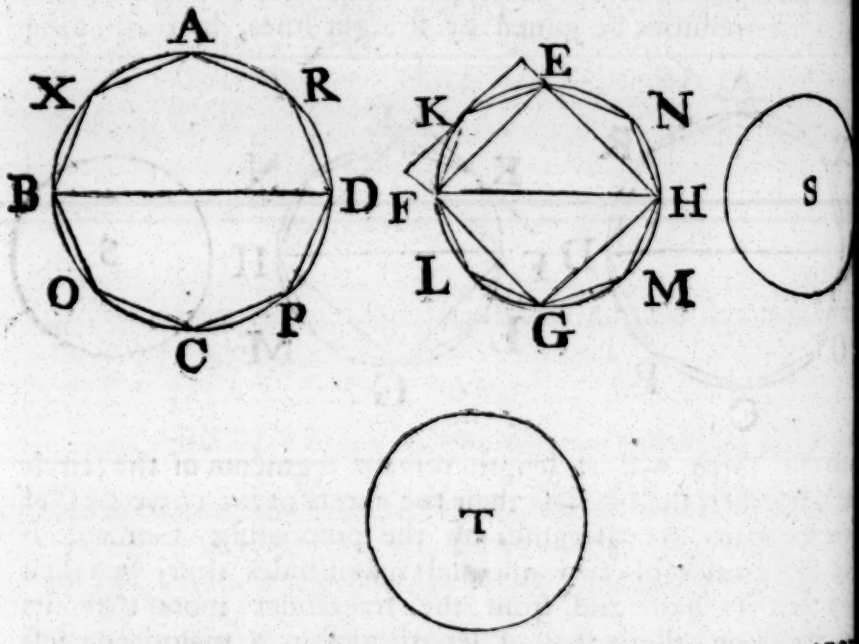
[a] 41. 1.



do this, there will at length remain ſegments of the circle which, together, ſhall be leſs than the exceſs of the circle EFGH above the ſpace S: Becauſe, by the preceeding Lemma, if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and ſo on, there ſhall at length remain a magnitude leſs than the leaſt of the propoſed magnitudes. Let then the ſegments EK, KF, FL, LG, GM, MH, HN, NE be thoſe that remain and are together leſs than the exceſs of the circle EFGH above S: Therefore the reſt of the circle, viz. the polygon EKFLGMHN, is greater than the ſpace S. Deſcribe likewiſe in the circle ABCD the polygon AXBOCPDR ſimilar to the polygon EKFLGMHN: As, therefore, the ſquare of BD is to the ſquare of FH, ſo [b] is the polygon AXBOCPDR to the polygon EKFLGMHN: But the ſquare of BD is alſo to the

[b] 1. 12.

Book XII. ſquare of FH, as the circle ABCD is to the ſpace S: There-
fore as the circle ABCD is to the ſpace S, ſo is [c] the polygon
[c] 11. 5. AXBOCPDR to the polygon EKFLGMHN: But the circle
ABCD is greater than the polygon contained in it; wherefore
[d] 14. 5. the ſpace S is greater [d] than the polygon EKFLGMHN: But
it is likewiſe leſs, as has been demonſtrated; which is impoſſi-
ble. Therefore the ſquare of BD is not to the ſquare of FH
as the circle ABCD is to any ſpace leſs than the circle EFGH.
In the ſame manner, it may be demonſtrated, that neither is the
ſquare of FH to the ſquare of BD, as the circle EFGH is to
any ſpace leſs than the circle ABCD. Nor is the ſquare of
BD to the ſquare of FH, as the circle ABCD is to any ſpace
greater than the circle EFGH: For, if poſſible, let it be ſo to
T, a ſpace greater than the circle EFGH: Therefore, inverſely,
as the ſquare of FH to the ſquare of BD, ſo is the ſpace T



the circle ABCD. But as the ſpace † T is to the circle ABCD,
ſo is the circle EFGH to ſome ſpace, which muſt be leſs [d] than
the circle ABCD, becauſe the ſpace T is greater, by hypothe-
ſis, than the circle EFGH. Therefore as the ſquare of FH is

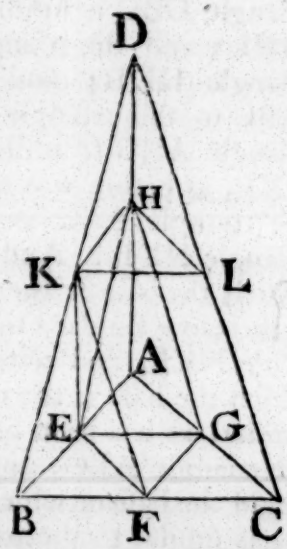
† For as, in the foregoing note, at *, it was explained how it was poſſible there could be a fourth proportional to the ſquares of BD, FH, and the circle ABCD, which was named S. So, in like manner, there can be a fourth proportional to this other ſpace, named T, and the circles ABCD, EFGH. And the like is to be underſtood in ſome of the following propoſitions.

e ſquare of BD, ſo is the circle EFGH to a ſpace leſs than e circle ABCD, which has been demonſtrated to be impoſble: Therefore the ſquare of BD is not to the ſquare of FH, the circle ABCD is to any ſpace greater than the circle FGH: And it has been demonſtrated that neither is the ſquare BD to the ſquare of FH, as the circle ABCD to any ſpace s than the circle EFGH: Wherefore, as the ſquare of BD to e ſquare of FH, ſo is the circle ABCD to the circle EFGH ‡. ircles therefore are, &c. Q. E. D. Book XII.

PROP. III. THEOR.

EVERY pyramid having a triangular baſe, may be divided into two equal and ſimilar pyramids aving triangular baſes, and which are ſimilar to the hole pyramid; and into two equal priſms which ogether are greater than half of the whole pyramid. See N.

Let there be a pyramid of which the baſe is the triangle ABC nd its vertex the point D: The pyramid ABCD may be divid into two equal and ſimilar pyramids aving triangular baſes, and ſimilar to e whole; and into two equal priſms hich together are greater than half of e whole pyramid.



Divide AB, BC, CA, AD, DB, DC, ch into two equal parts in the points , F, G, H, K, L, and join EH, EG, H, HK, KL, LH, EK, KF, FG. Beuſe AE is equal to EB, and AH to D, HE is parallel [a] to DB: For the me reaſon, HK is parallel to AB: herefore HEBK is a parallelogram, and K equal [b] to EB: But EB is equal to E; therefore alſo AE is equal to K: And AH is equal to HD; wherere EA, AH are equal to KH, HD, ch to each; and the angle EAH is equal [c] to the angle KHD; herefore the baſe EH is equal to the baſe KD, and the triangle AEH

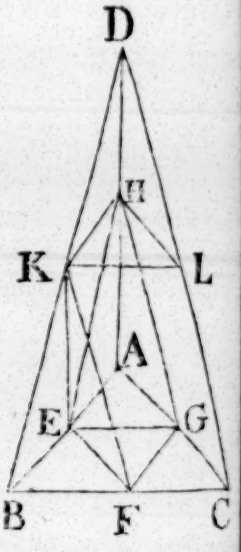
[a] 2. 6.

[b] 34. 1.

[c] 29. 1.

‡ Becauſe as a fourth proportional to the ſquares of BD, FH and the circle ABCD poſſible, and that it can neither be leſs nor greater than the circle EFGH, it muſt equal to it.

Book XII. AEH equal [d] and similar to the triangle HKD: For the same
[d] 4. 1. reason, the triangle AGH is equal and similar to the triangle
HLD: And because the two straight lines EH, HG which
meet one another are parallel to KD, DL that meet one another, and are not in the same plane with them, they contain
[e] 10. 11. equal [e] angles; therefore the angle EHG is equal to the angle KDL. Again, because EH, HG are equal to KD, DL, each
to each, and the angle EHG equal to the angle KDL; therefore the base EG is equal to the base KL: And the triangle
EHG equal [d] and similar to the triangle KDL: For the same
reason, the triangle AEG is also equal and similar to the triangle HKL. Therefore the pyramid of which the base is the
triangle AEG, and of which the vertex is the point H, is
[f] C. 11. equal [f] and similar to the pyramid the base of which is the triangle KHL, and vertex the point D: And because HK is parallel to AB a side of the triangle ADB, the triangle ADB is equiangular to the triangle HDK, and their
[g] 4. 6. sides are proportionals [g]: Therefore the triangle ADB is similar to the triangle HDK: And for the same reason, the triangle DBC is similar to the triangle DKL; and the triangle ADC to the triangle HDL; and also the triangle ABC to the triangle AEG: But the triangle AEG is similar to the triangle HKL, as before was proved; therefore
[h] 21. 6. the triangle ABC is similar [h] to the triangle HKL: And the pyramid of which the base is the triangle ABC, and vertex the point D,
[i] B. 11. & 11. Def. 11. is therefore similar [i] to the pyramid of which the base is the triangle HKL, and vertex the same point D: But the pyramid of which the base is the triangle HKL, and vertex the point D, is similar, as has been proved, to the pyramid the base of which is the triangle AEG, and vertex the point H: Wherefore the pyramid the base of which is the triangle ABC, and vertex the point D, is similar to the pyramid of which the base is the triangle AEG and vertex H: Therefore each of the pyramids AEGH, HKLD is similar to the whole pyramid ABCD: And because
[k] 41. 1. BF is equal to FC, the parallelogram EBFG is double [k] of the triangle GFC: But when there are two prisms of the same al-



tude, of which one has a parallelogram for its baſe, and the other a triangle that is half of the parallelogram, theſe priſms are equal to one another; therefore the priſm having the parallelogram EBFG for its baſe, and the ſtraight line KH oppoſite to it, is equal to the priſm having the triangle GFC for its baſe, and the triangle HKL oppoſite to it; for they are of the ſame altitude, becauſe they are between the parallel planes ABC, HKL: And it is manifeſt that each of theſe priſms is greater than either of the pyramids of which the triangles AEG, HKL are the baſes, and the vertices the points H, D; becauſe, if EF be joined, the priſm having the parallelogram EBFG for its baſe, and KH the ſtraight line oppoſite to it, is greater than the pyramid of which the baſe is the triangle EBF, and vertex the point K; but this pyramid is equal to the pyramid the baſe of which is the triangle AEG, and vertex the point H; becauſe they are contained by equal and ſimilar planes: Wherefore the priſm having the parallelogram EBFG for its baſe, and oppoſite ſide KH, is greater than the pyramid of which the baſe is the triangle AEG, and vertex the point H: And the priſm of which the baſe is the parallelogram EBFG, and oppoſite ſide KH is equal to the priſm having the triangle GFC for its baſe, and HKL the triangle oppoſite to it; and the pyramid of which the baſe is the triangle AEG, and vertex H, is equal to the pyramid of which the baſe is the triangle HKL, and vertex D: Therefore the two priſms before mentioned are greater than the two pyramids of which the baſes are the triangles AEG, HKL, and vertices the points H, D. Therefore the whole pyramid of which the baſe is the triangle ABC, and vertex the point D, is divided into two equal pyramids ſimilar to one another, and to the whole pyramid; and into two equal priſms; and the two priſms are together greater than half of the whole pyramid. Q. E. D.

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[a] 40. 11.

[b] 15. 11.

[c] C. 11.

PROP.

Book. XII.

PROP. IV. THEOR.

See N. IF there be two pyramids of the ſame altitude, up on triangular baſes, and each of them be divide into two equal pyramids ſimilar to the whole pyra mid, and alſo into two equal priſms; and if each o theſe pyramids be divided in the ſame manner as th firſt two, and ſo on: As the baſe of one of the firſ two pyramids is to the baſe of the other, ſo ſhall a the priſms in one of them be to all the priſms in th other, that are produced by the ſame number of di viſions.

Let there be two pyramids of the ſame altitude upon the tri angular baſes ABC, DEF, and having their vertices in th points G, H; and let each of them be divided into two equ pyramids ſimilar to the whole, and into two equal priſms; an let each of the pyramids thus made be conceived to be divide in the like manner, and ſo on: As the baſe ABC is to the ba DEF, ſo are all the priſms in the pyramid ABCG to all th priſms in the pyramid DEFH made by the ſame number of di viſions.

Make the ſame conſtruction as in the foregoing propoſition and becauſe BX is equal to XC, and AL to LC, therefore X
[a] 2. 6. is parallel [a] to AB, and the triangle ABC ſimilar to the tri angle LXC: For the ſame reaſon, the triangle DEF is ſimila to RVF: And becauſe BC is double of CX, and EF double o FV, therefore BC is to CX, as EF to FV: And upon BC, C are deſcribed the ſimilar and ſimilarly ſituated rectilineal f gures ABC, LXC; and upon EF, FV, in like manner, a deſcribed the ſimilar figures DEF, RVF: Therefore, as the tri
[b] 22. 6. angle ABC is to the triangle LXC, ſo [b] is the triangle DEF t the triangle RVF, and, by permutation, as the triangle AB to the triangle DEF, ſo is the triangle LXC to the triang RVF: And becauſe the planes ABC, OMN, as alſo the plan
[c] 15. 11. DEF, STY are parallel [c], the perpendiculars drawn from th points G, H to the baſes ABC, DEF, which, by the hypoth ſis, are equal to one another, ſhall be cut each into two equ
[d] 17. 11. [d] parts by the planes OMN, STY, becauſe the ſtraight lin GC, HF are cut into two equal parts in the points N, Y the ſame planes: Therefore the priſms LXCOMN, RVFST are of the ſame altitude; and therefore, as the baſe LXC th

e baſe RVF; that is, as the triangle ABC to the triangle EF, ſo [a] is the priſm having the triangle LXC for its baſe, nd OMN the triangle oppoſite to it, to the priſm of which the aſe is the triangle RVF, and the oppoſite triangle STY: And ecauſe the two priſms in the pyramid ABCG are equal to he another, and alſo the two priſms in the pyramid DEFH qual to one another, as the priſm of which the baſe is the pallelogram KBXL and oppoſite ſide MO, to the priſm having he triangle LXC for its baſe, and OMN the triangle oppoſite o it; ſo is the priſm of which the baſe [b] is the parallelogram PEVR, and oppoſite ſide TS, to the priſm of which the baſe the triangle RVF, and oppoſite triangle STY. Therefore, omponendo, as the priſms KBXLMO, LXCOMN together

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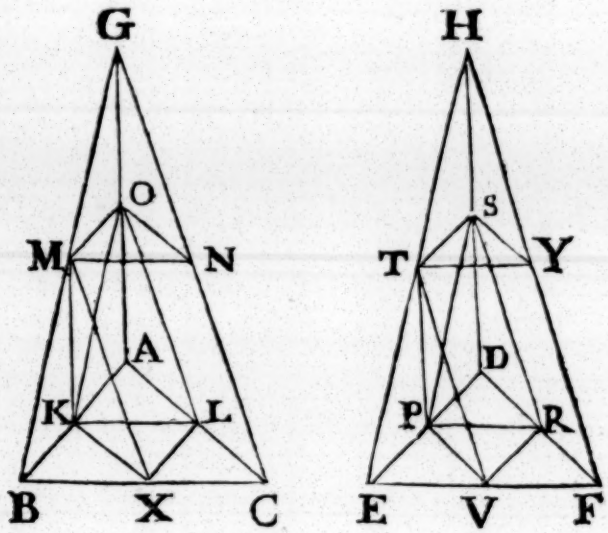
[a] Cor. 32. 11.
[b] 7. 5.



re unto the priſm LXCOMN; ſo are the priſms PEVRTS, RVFSTY to the priſm RVFSTY: And, permutando, as the riſms KBXLMO, LXCOMN are to the priſms PEVRTS, RVFSTY; ſo is the priſm LXCOMN to the priſm RVFSTY: But as the priſm LXCOMN to the priſm RVFSTY, ſo is, as has been proved, the baſe ABC to the baſe DEF: Therefore, as the baſe ABC to the baſe DEF, ſo are the two priſms in he pyramid ABCG to the two priſms in the pyramid DEFH: And likewiſe if the pyramids now made, for example, the two OMNG, STYH be divided in the ſame manner; as the baſe OMN is to the baſe STY, ſo ſhall the two priſms in the pyramid OMNG be to the two priſms in the pyramid STYH: But the baſe OMN is to the baſe STY, as the baſe ABC to the baſe DEF; therefore, as the baſe ABC to the baſe DEF, ſo are the

Book XII. the two prisms in the pyramid ABCG to the two prisms in the pyramid DEFH; and so are the two prisms in the pyramid OMNG to the two prisms in the pyramid STYH; and so are all four to all four: And the same thing may be shewn of the prisms made by dividing the pyramids AKLO and DPRS, and of all made by the same number of divisions. Q. E. D.

PROP. V. THEOR.

See N. PYRAMIDS of the same altitude which have triangular bases, are to one another as their bases.

Let the pyramids of which the triangles ABC, DEF are the bases, and of which the vertices are the points G, H, be of the same altitude: As the base ABC to the base DEF, so is the pyramid ABCG to the pyramid DEFH.

For, if it be not so, the base ABC must be to the base DEF as the pyramid ABCG to a solid either less than the pyramid DEFH, or greater than it ‡. First, let it be to a solid less than it, viz. to the solid Q: And divide the pyramid DEFH into two equal pyramids, similar to the whole, and into two equal prisms: Therefore these two prisms are greater [a] than the half of the whole pyramid. And, again, let the pyramids made by this division be in like manner divided, and so on, until the pyramids which remain undivided in the pyramid DEFH be all of them together, less than the excess of the pyramid DEFH above the solid Q: Let these, for example, be the pyramids DPRS, STYH: Therefore the prisms, which make the rest of the pyramid DEFH, are greater than the solid Q: Divide likewise the pyramid ABCG in the same manner, and into as many parts, as the pyramid DEFH: Therefore, as the base ABC to the base DEF, so [b] are the prisms in the pyramid ABCG to the prisms in the pyramid DEFH: But as the base ABC to the base DEF, so, by hypothesis, is the pyramid ABCG to the solid Q; and therefore, as the pyramid ABCG to the solid Q, so are the prisms in the pyramid ABCG to the prisms in the pyramid DEFH: But the pyramid ABCG is greater than the prisms contained in it; wherefore [c] also the solid Q is greater than the prisms in the pyramid DEFH. But it is also less, which is impossible. Therefore the base ABC is not to the

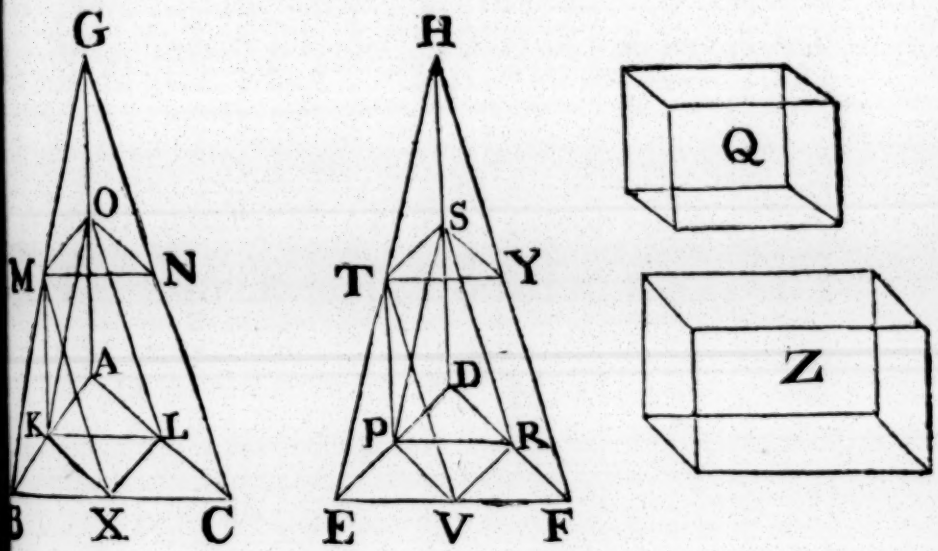
[a] 3. 12.

[b] 4. 12.

[c] 14. 5.

‡ This may be explained the same way as at the note * in proposition 2. in the like case.

base DEF, as the pyramid ABCG to any ſolid which is Book XII.
than the pyramid DEFH. In the ſame manner it may
demonſtrated, that the baſe DEF is not to the baſe ABC,
the pyramid DEFH to any ſolid which is leſs than the py-
mid ABCG. Nor can the baſe ABC be to the baſe DEF, as
pyramid ABCG to any ſolid which is greater than the py-
mid DEFH. For, if it be poſſible, let it be ſo to a greater,
the ſolid Z. And becauſe the baſe ABC is to the baſe DEF,
the pyramid ABCG to the ſolid Z; by inverſion, as the baſe
EF to the baſe ABC, ſo is the ſolid Z to the pyramid ABCG,
t as the ſolid Z is to the pyramid ABCG, ſo is the pyramid



EFH to ſome ſolid ‡, which muſt be leſs [a] than the pyramid [a] 14. 5.
BCG, becauſe the ſolid Z is greater than the pyramid DEFH.
nd therefore, as the baſe DEF to the baſe ABC, ſo is the py-
mid DEFH to a ſolid leſs than the pyramid ABCG; the con-
ry to which has been proved. Therefore the baſe ABC is
t to the baſe DEF, as the pyramid ABCG to any ſolid which
greater than the pyramid DEFH. And it has been proved
at neither is the baſe ABC to the baſe DEF, as the pyramid
BCG to any ſolid which is leſs than the pyramid DEFH.
herefore, as the baſe ABC is to the baſe DEF, ſo is the py-
mid ABCG to the pyramid DEFH. Wherefore pyramids,
c. Q. E. D.

PROP.

‡ This may be explained the ſame way as the like at the mark † in prop 2.

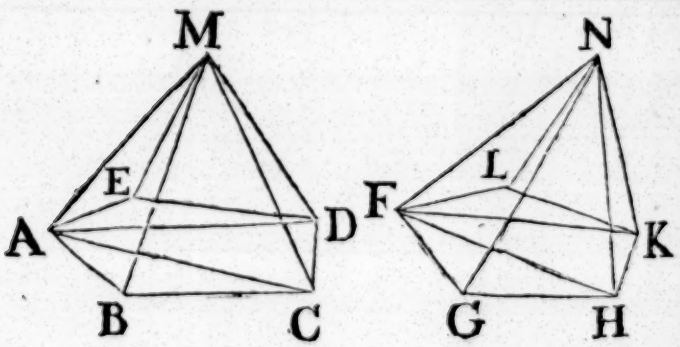
PROP. VI. THEOR.

See N. PYRAMIDS of the same altitude which have polygons for their bases, are to one another as their bases.

Let the pyramids which have the polygons ABCDE, FGHKL for their bases, and their vertices in the points M, N, be of the same altitude: As the base ABCDE to the base FGHKL, so the pyramid ABCDEM to the pyramid FGHKLN.

Divide the base ABCDE into the triangles ABC, ACD, ADE; and the base FGHKL into the triangles FGH, FHK, FKL: And upon the bases ABC, ACD, ADE let there be as many pyramids of which the common vertex is the point M, and upon the remaining bases as many pyramids having their common vertex in the point N: Therefore, since the triangle ABC is to the triangle FGH, as [a] the pyramid ABCM to the pyramid FGHN; and the triangle ACD to the triangle FGH as the pyramid ACDM to the pyramid FGHN; and also

[a] 5. 12.



triangle ADE to the triangle FGH, as the pyramid ADEM to the pyramid FGHN; as all the first antecedents to their common consequent; so [b] are all the other antecedents to their common consequent; that is, as the base ABCDE to the base FGH, so is the pyramid ABCDEM to the pyramid FGHKLN. And for the same reason, as the base FGHKL to the base FGH, so is the pyramid FGHKLN to the pyramid FGHN: And, inversion, as the base FGH to the base FGHKL, so is the pyramid FGHN to the pyramid FGHKLN: Then, because as the base ABCDE to the base FGH, so is the pyramid ABCDEM to the pyramid FGHN; and as the base FGH to the base FGHKL, so is the pyramid FGHN to the pyramid FGHKLN

[b] 2. Cor. 24. 5.

therefore

erefore, ex aequali [c], as the baſe ABCDE to the baſe FGHKL, the pyramid ABCDEM to the pyramid FGHKLN. Therefore pyramids, &c. Q. E. D.

Book XII.

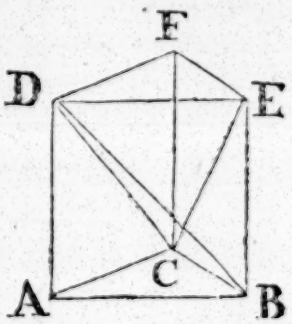
[c] 22. 5.

PROP. VII. THEOR.

EVERY priſm having a triangular baſe may be divided into three pyramids that have triangular ſes, and are equal to one another.

Let there be a priſm of which the baſe is the triangle ABC, d let DEF be the triangle oppoſite to it: The priſm ABCDEF y be divided into three equal pyramids having triangular ſes.

Join BD, EC, CD; and becauſe ABED is a parallelogram which BD is the diameter, the triangle ABD is equal [a] to triangle EBD; therefore the pyramid of which the baſe is triangle ABD, and vertex the point C, is equal [b] to the ramid of which the baſe is the triangle EBD, and vertex the int C: But this pyramid is the ſame with the pyramid the ſe of which is the triangle EBC, and vertex the point D; they are contained by the ſame planes: Therefore the pymid of which the baſe is the triangle ABD, and vertex the int C, is equal to the pyramid the baſe of which is the trigle EBC, and vertex the point D: Again, becauſe FCBE is parallelogram of which the diameter is E, the triangle ECF is equal [a] to the angle ECB; therefore the pyramid of hich the baſe is the triangle ECB, and rtex the point D, is equal to the pyraid the baſe of which is the triangle CF, and vertex the point D: But the ramid of which the baſe is the triangle CB, and vertex the point D has been oved equal to the pyramid of which the ſe is the triangle ABD, and vertex the point C. Therefore e priſm ABCDEF is divided into three equal pyramids having angular baſes, viz. into the pyramids ABDC, EBDC, ECFD: nd becauſe the pyramid of which the baſe is the triangle ABD, d vertex the point C, is the ſame with the pyramid of which e baſe is the triangle ABC, and vertex the point D, for they e contained by the ſame planes; and that the pyramid of which e baſe is the triangle ABD, and vertex the point C, has been



[a] 34. 1.

[b] 5. 12.

Book XII. demonſtrated to be a third part of the priſm the baſe of whic
is the triangle ABC, and to which DEF is the oppoſite tr
angle; therefore the pyramid of which the baſe is the triang
ABC, and vertex the point D, is the third part of the pri
which has the ſame baſe, viz. the triangle ABC, and DEF
the oppoſite triangle. Q. E. D.

Cor. 1. From this it is manifeſt, that every pyramid is t
third part of a priſm which has the ſame baſe, and is of an
qual altitude with it; for, if the baſe of the priſm be any oth
figure than a triangle, it may be divided into priſms having t
angular baſes.

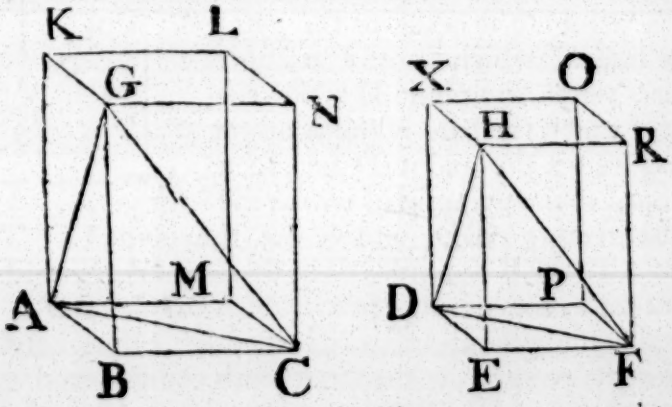
Cor. 2. Priſms of equal altitudes are to one another as the
baſes; becauſe the pyramids upon the ſame baſes, and of t
[c] 6. 12. ſame altitude, are [c] to one another as their baſes.

PROP. VIII. THEOR.

**Similar pyramids having triangular baſes are on
to another in the triplicate ratio of that of the
homologous ſides.**

Let the pyramids having the triangles ABC, DEF for th
baſes, and the points G, H for their vertices, be ſimilar and
milarly ſituated; the pyramid ABCG has to the pyramid DEF
the triplicate ratio of that which the ſide BC has to the homo
gous ſide EF.

Complete the parallelograms ABCM, GBCN, ABGK, a
the ſolid parallelepiped BGML contained by theſe planes a



theſe oppoſite to them: And, in like manner, complete the
lid parallelepiped EHPO contained by the three parallelogra
DEFP, HEFR, DEHX, and thoſe oppoſite to them: And
cauſe the pyramid ABCG is ſimilar to the pyramid DEFH,

ngle ABC is equal [a] to the angle DEF, and the angle GBC Book XII.
the angle HEF, and ABG to DEH: And AB is [b] to BC,
DE to EF; that is, the sides about the equal angles are pro- [a] 11. def.
portionals; wherefore the parallelogram BM is similar to EP: 11.
For the same reason, the parallelogram BN is similar to ER, [b] 1. def.
nd BK to EX: Therefore the three parallelograms BM, BN, 6.
K are similar to the three EP, ER, EX: But the three BM,
N, BK, are equal and similar [c] to the three which are oppo- [c] 24. 11.
te to them, and the three EP, ER, EX equal and similar to
e three opposite to them: Wherefore the solids BGML,
HPO are contained by the same number of similar planes;
nd their solid angles are equal [d]; and therefore the solid [d] B. 11.
GML is similar [a] to the solid EHPO: But similar solid pa-
llelepipeds have the triplicate [e] ratio of that which their ho- [e] 33. 11.
mologous sides have: Therefore the solid BGML has to the
lid EHPO the triplicate ratio of that which the side BC has
the homologous side EF: But as the solid BGML is to the
lid EHPO, so is [f] the pyramid BCG to the pyramid DEFH; [f] 15. 5.
ecause the pyramids are the sixth part of the solids, since the
rism, which is the half [g] of the solid parallelepiped, is triple [h] [g] 28. 11.
f the pyramid. Wherefore likewise the pyramid ABCG has [h] 7. 12.
the pyramid DEFH, the triplicate ratio of that which BC has
the homologous side EF. Q. E. D.

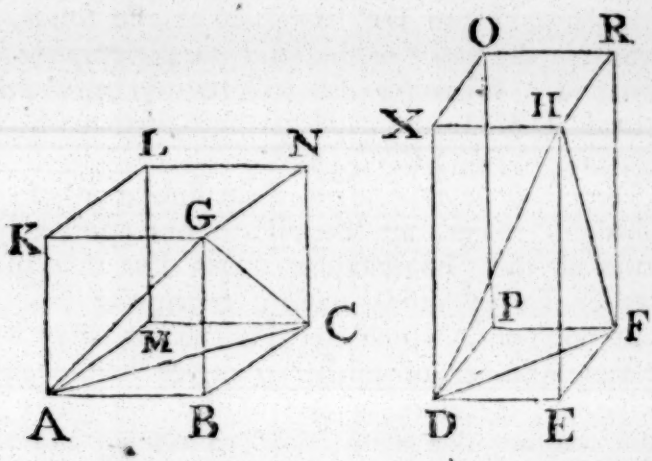
Cor. From this it is evident, that similar pyramids which See N.
ave multangular bases, are likewise to one another in the tri-
licate ratio of their homologous sides: For they may be di-
ided into similar pyramids having triangular bases, because
he similar polygons, which are their bases, may be divided
nto the same number of similar triangles homologous to the
whole polygons; therefore as one of the triangular pyramids
n the first multangular pyramid is to one of the triangular
yramids in the other, so are all the triangular pyramids in the
rst to all the triangular pyramids in the other; that is, so is
he first multangular pyramid to the other: But one triangu-
r pyramid is to its similar triangular pyramid, in the tripli-
ate ratio of their homologous sides; and therefore the first
ultangular pyramid has to the other, the triplicate ratio of
hat which one of the sides of the first has to the homologous
de of the other.

PROP. IX. THEOR.

THE baſes and altitudes of equal pyramids having triangular baſes are reciprocally proportional: And triangular pyramids of which the baſes and altitudes are reciprocally proportional, are equal to one another.

Let the pyramids of which the triangles ABC, DEF are the baſes, and which have their vertices in the points G, H, be equal to one another: The baſes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional, viz. the baſe ABC is to the baſe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG.

Complete the parallelograms AC, AG, GC, DF, DH, HE, and the ſolid parallelepipeds BGML, EHPO contained



theſe planes and thoſe oppoſite to them: And becauſe the pyramid ABCG is equal to the pyramid DEFH, and that the ſolid BGML is ſextuple of the pyramid ABCG, and the ſolid EHPO ſextuple of the pyramid DEFH; therefore the ſolid
[a] 1. Ax. 5. BGML is equal [a] to the ſolid EHPO: But the baſes and altitudes of equal ſolid parallelepipeds are reciprocally proportional [b]; therefore as the baſe BM to the baſe EP, ſo is the altitude of the ſolid EHPO to the altitude of the ſolid BGML.
[b] 34. 11.
[c] 15. 5. But as the baſe BM to the baſe EP, ſo is [c] the triangle ABC to the triangle DEF; therefore as the triangle ABC to the triangle DEF, ſo is the altitude of the ſolid EHPO to the altitude of the ſolid BGML: But the altitude of the ſolid EHPO is the ſame with the altitude of the pyramid DEFH; and the altitude of the ſolid BGML is the ſame with the altitude of the

pyram

ramid ABCG: Therefore, as the baſe ABC to the baſe DEF, Book XII
is the altitude of the pyramid DEFH to the altitude of the
ramid ABCG: Wherefore the baſes and altitudes of the py-
mids ABCG, DEFH are reciprocally proportional.

Again, Let the baſes and altitudes of the pyramids ABCG,
EFH be reciprocally proportional, viz. the baſe ABC to the
ſe DEF, as the altitude of the pyramid DEFH to the alti-
de of the pyramid ABCG: The pyramid ABCG is equal to
e pyramid DEFH.

The ſame conſtruction being made, becauſe as the baſe ABC
the baſe DEF, ſo is the altitude of the pyramid DEFH to
e altitude of the pyramid ABCG; and as the baſe ABC to
e baſe DEF, ſo is the parallelogram BM to the parallelo-
am EP; therefore the parallelogram BM is to EP, as the
itude of the pyramid DEFH to the altitude of the pyramid
BCG: But the altitude of the pyramid DEFH is the ſame
ith the altitude of the ſolid parallelepiped EHPO; and the al-
tude of the pyramid ABCG is the ſame with the altitude of
e ſolid parallelepiped BGML: As, therefore, the baſe BM to
e baſe EP, ſo is the altitude of the ſolid parallelepiped EHPO
the altitude of the ſolid parallelepiped BGML. But ſolid
rallelepipeds having their baſes and altitudes reciprocally
oportional, are equal [b] to one another. Therefore the ſolid [b] 34. 11.
rallelepiped BGML is equal to the ſolid parallelepiped EHBO.
nd the pyramid ABCG is the ſixth part of the ſolid BGML,
d the pyramid DEFH the ſixth part of the ſolid EHPO.
herefore the pyramid ABCG is equal to the pyramid DEFH.
herefore the baſes, &c. Q. E. D.

PROP. X. THEOR.

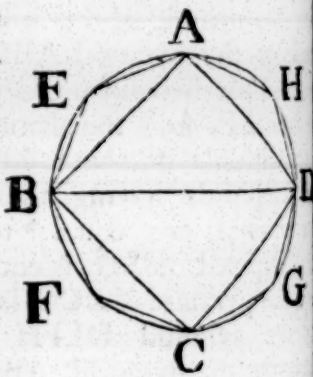
EVERY cone is the third part of a cylinder which has the ſame baſe, and is of an equal altitude with it.

Let a cone have the ſame baſe with a cylinder, viz. the cir-
e ABCD, and the ſame altitude. The cone is the third part
the cylinder; that is, the cylinder is triple of the cone.

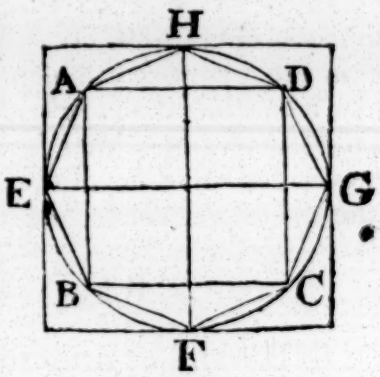
If the cylinder be not triple of the cone, it muſt either be
eater than the triple, or leſs than it. Firſt, Let it be great-
than the triple; and deſcribe the ſquare ABCD in the cir-
e; this ſquare is greater than the half of the circle ABCD †.

† As was ſhewn in prop. 2. of this book.

Book XII. Upon the ſquare ABCD erect a priſm of the ſame altitude wi
the cylinder; this priſm is greater than half of the cylinder
becauſe, if a ſquare be deſcribed about the circle, and a pri
erected upon the ſquare, of the ſame altitude with the cylinde
the inſcribed ſquare is half of that circumſcribed; and up
theſe ſquare baſes are erected ſolid parallelepipeds, viz. t
priſms, of the ſame altitude; therefore the priſm upon t
ſquare ABCD is the half of the priſm upon the ſquare deſcr
bed about the circle; becauſe they are to one another as the
[a] 32. 11. baſes [a]: And the cylinder is leſs than the priſm upon the ſquar
deſcribed about the circle ABCD: Therefore the priſm upo
the ſquare ABCD of the ſame altitude with the cylinder,
greater than half of the cylinder. Biſect the circumferenc
AB, BC, CD, DA in the points E, F, G, H; and join A
EB, BF, FC, CG, GD, DH, HA: Then, each of the triangl
AEB, BFC, CGD, DHA is greater than the half of the ſeg
ment of the circle in which it ſtands, as was ſhewn in Prop. 2. of this Book. Erect priſms upon each of theſe triangles of the ſame altitude with the cylinder; each of theſe priſms is greater than half of the ſegment of the cylinder in which it is; becauſe, if, through the points E, F, G, H, parallels be drawn to AB, BC, CD, DA, and parallelograms be completed upon the ſame AB, BC, CD, DA, and ſolid parallelepipeds be erected upon the parallelograms; the priſms upon t
triangles AEB, BFC, CGD, DHA are the halves of the ſo
[b] 2. Cor. 7. 12. lid parallelepipeds [b]. And the ſegments of the cylinder whi
are upon the ſegments of the circle cut off by AB, BC, C
DA, are leſs than the ſolid parallelepipeds which contain the
Therefore the priſms upon the triangles AEB, BFC, CG
DHA, are greater than half of the ſegments of the cylinder
which they are; therefore, if each of the circumferences be d
vided into two equal parts, and ſtraight lines be drawn fro
the points of diviſion to the extremities of the circumferenc
and upon the triangles thus made priſms be erected of the ſam
altitude with the cylinder, and ſo on, there muſt at length re
[c] Lemma. main ſome ſegments of the cylinder which together are leſs
than the exceſs of the cylinder above the triple of the con
Let them be thoſe upon the ſegments of the circle AE, EB, BF, FC

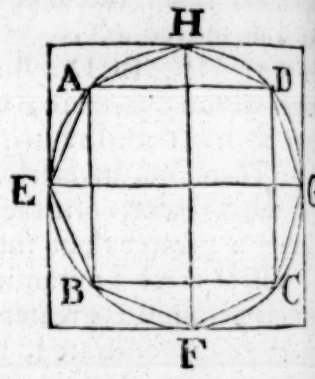


, CG, GD, DH, HA. Therefore the reſt of the cylin- Book XII.
that is, the priſm of which the baſe is the polygon
BFCGDH, and of which the altitude is the ſame with that
he cylinder, is greater than the triple of the cone: But this
m is triple [d] of the pyramid upon the ſame baſe, of which [d] 1. Cor. 7. 12.
vertex is the ſame with the vertex of the cone; therefore
pyramid upon the baſe AEBFCGDH, having the ſame
ex with the cone, is greater than the cone, of which the
e is the circle ABCD: But it is alſo leſs, for the pyramid is
tained within the cone; which is impoſſible. Nor can the
nder be leſs than the triple of the cone. Let it be leſs, if
ible: Therefore, inverſely, the cone is greater than the third
t of the cylinder. In the circle ABCD deſcribe a ſquare;
ſquare is greater than the half of the circle: And upon the
are ABCD erect a pyramid having the ſame vertex with the
e; this pyramid is greater than the half of the cone; becauſe,
was before demonſtrated, if a ſquare be deſcribed about the
le, the ſquare ABCD is the
of it; and if, upon theſe ſquares,
re be erected ſolid parallelepi-
s of the ſame altitude with the
e, which are alſo priſms, the
m upon the ſquare ABCD
ll be the half of that which is
n the ſquare deſcribed about the
le; for they are to one another
heir baſes [a]; as are alſo the third
s of them: Therefore the py-
id, the baſe of which is the
are ABCD, is half of the pyramid upon the ſquare de-
bed about the circle: But this laſt pyramid is greater than
cone which it contains; therefore the pyramid upon the
are ABCD having the ſame vertex with the cone, is great-
han the half of the cone. Biſect the circumferences AB,
, CD, DA in the points E, F, G, H, and join AE, EB,
, FC, CG, GD, DH, HA: Therefore each of the triangles
B, BFC, CGD, DHA is greater than half of the ſegment
he circle in which it is: Upon each of theſe triangles erect
amids having the ſame vertex with the cone. Therefore each
heſe pyramids is greater than the half of the ſegment of
cone in which it is, as before was demonſtrated of the
ms and ſegments of the cylinder; and thus dividing each
he circumferences into two equal parts, and joining the



[a] 32. 11.

Book XII. points of diviſion and their extremities by ſtraight lines, a upon the triangles erecting pyramids having their vertices t ſame with that of the cone, and ſo on, there muſt at leng remain ſome ſegments of the cone, which together ſhall be le than the exceſs of the cone above the third part of the cyli der. Let theſe be the ſegments upon AE, EB, BF, FC, CG, G DH, HA. Therefore the reſt of the the cone, that is, the pyramid, of which the baſe is the polygon AEBFCGDH, and of which the vertex is the ſame with that of the cone, is greater than the third part of the cylinder. But this pyramid is the third part of the priſm upon the ſame baſe AEBFCGDH, and of the ſame altitude with the cylinder. Therefore this priſm is greater than the cylinder of which the baſe is the circle ABCD. But it is alſo leſs, for it is contain within the cylinder; which is impoſſible. Therefore the c linder is not leſs than the triple of the cone. And it has be demonſtrated that neither is it greater than the triple. Therefo the cylinder is triple of the cone, or, the cone is the third pa of the cylinder. Wherefore every cone, &c. Q. E. D.



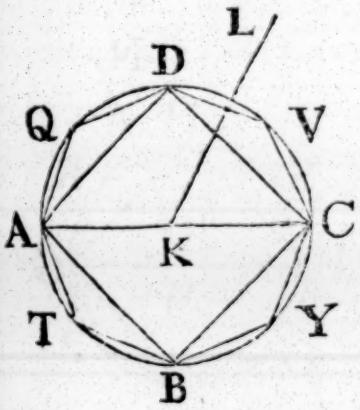
PROP. XI. THEOR.

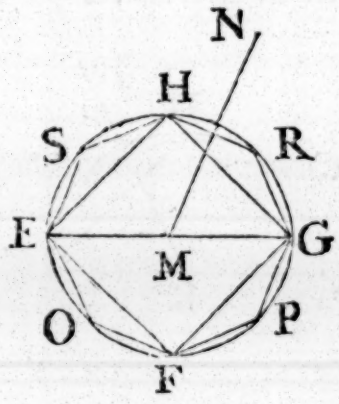
See N. CONES and cylinders of the ſame altitude, are t one another as their baſes.

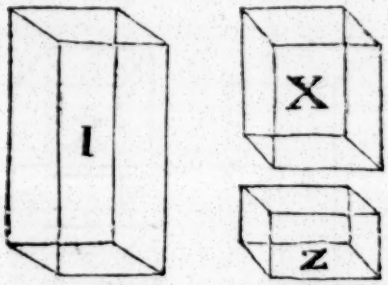
Let the cones and cylinders, of which the baſes are the ci cles ABCD, EFGH, and the axes KL, MN, and AC, EG t diameters of their baſes, be of the ſame altitude. As the circ ABCD to the circle EFGH, ſo is the cone AL to the cone E

If it be not ſo, let the circle ABCD be to the circle EFG as the cone AL to ſome ſolid either leſs than the cone EN, greater than it. Firſt, let it be to a ſolid leſs than EN, viz. the ſolid X; and let Z be the ſolid which is equal to the e ceſs of the cone EN above the ſolid X; therefore the cone E is equal to the ſolids X, Z together. In the circle EFGH d ſcribe the ſquare EFGH, therefore this ſquare is greater th the half of the circle: Upon the ſquare EFGH erect a pyr mid of the ſame altitude with the cone; this pyramid greater than half of the cone. For, if a ſquare be deſcrib about the circle, and a pyramid be erected upon it,

ng the ſame vertex with the cone †, the pyramid inſcribed Book XII.
the cone is half of the pyramid circumſcribed about it, be-
uſe they are to one another as their baſes [a]: But the cone is [a] 6. 12.
ſs than the circumſcribed pyramid; therefore the pyramid of
hich the baſe is the ſquare EFGH, and its vertex the ſame
ith that of the cone, is greater than half of the cone: Divide
e circumferences EF, FG, GH, HE, each into two equal
rts in the points, O, P, R, S, and join EO, OF, FP, PG,
R, RH, HS, SE: Therefore each of the triangles EOF,
PG, GRH, HSE is greater than half of the ſegment of the







rcle in which it is: Upon each of theſe triangles erect a pyra-
id having the ſame vertex with the cone; each of theſe py-
mids is greater than the half of the ſegment of the cone in
hich it is: And thus dividing each of theſe circumferences in-
two equal parts, and from the points of diviſion drawing
raight lines to the extremities of the circumferences, and
on each of the triangles thus made erecting pyramids having
e ſame vertex with the cone, and ſo on, there muſt at length
main ſome ſegments of the cone which are together leſs [b] [b] Lemma.
an the ſolid Z: Let theſe be the ſegments upon EO, OF, FP,
PG,

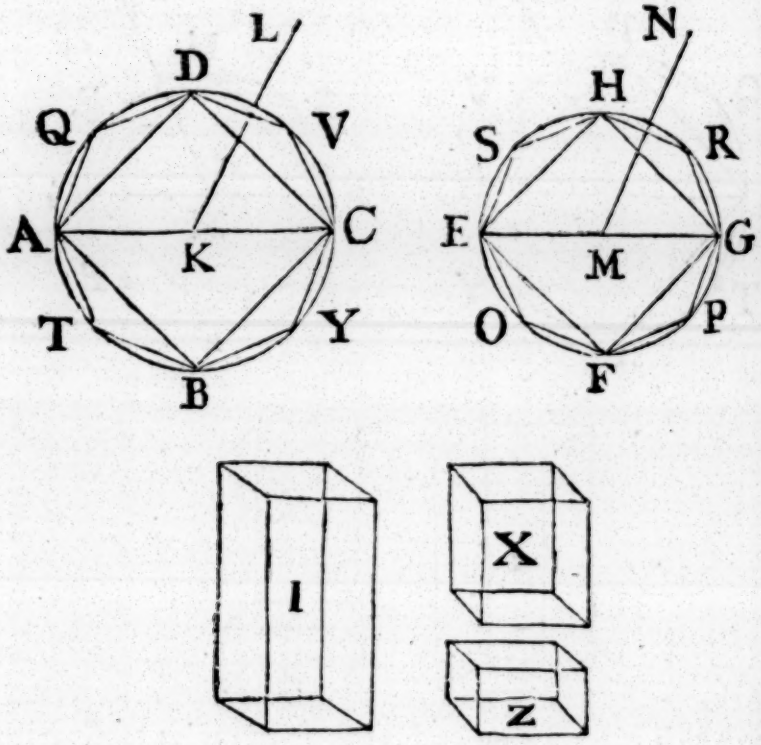
† Vertex is put in place of altitude which is in the Greek, becauſe the pyra-
id, in what follows, is ſuppoſed to be circumſcribed about the cone, and ſo muſt
ve the ſame vertex. And the ſame change is made in ſome places following.

Book XII. PG, GR, RH, HS, SE: Therefore the remainder of the cone, viz. the pyramid of which the baſe is the polygon EOFPGRHS, and its vertex the ſame with that of the cone, is greater than the ſolid X: In the circle ABCD deſcribe the polygon ATBYCVDQ ſimilar to the polygon EOFPGRHS, and upon it erect a pyramid having the ſame vertex with the cone AL.

[a] 1. 12. And becauſe as the ſquare of AC is to the ſquare of EG, ſo [a] is the polygon ATBYCVDQ to the polygon EOFPGRHS; and

[b] 2. 12. as the ſquare of AC to the ſquare of EG, ſo is [b] the circle

[c] 11. 5. ABCD to the circle EFGH; therefore the circle ABCD is [c] to the circle EFGH, as the polygon ATBYCVDQ to the poly-



gon EOFPGRHS: But as the circle ABCD to the circle EFGH, ſo is the cone AL to the ſolid X; and as the polygon

[d] 6. 12. ATBYCVDQ to the polygon EOFPGRHS, ſo is [d] the pyramid of which the baſe is the firſt of thoſe polygons, and vertex L, to the pyramid of which the baſe is the other polygon, and its vertex N: Therefore, as the cone AL to the ſolid X, ſo is the pyramid of which the baſe is the polygon ATBYCVDQ, and vertex L, to the pyramid the baſe of which is the polygon EOFPGRHS, and vertex N: But the cone AL is greater than

[e] 14. 5. the pyramid contained in it; therefore the ſolid X is greater [e] than the pyramid in the cone EN: But it is leſs, as was ſhewn;

which

hich is abſurd: Therefore the circle ABCD is not to the cir- Book XII.
e EFGH, as the cone AL to any ſolid which is leſs than the
ne EN. In the ſame manner it may be demonſtrated that
e circle EFGH is not to the circle ABCD, as the cone EN
any ſolid leſs than the cone AL. Nor can the circle ABCD
to the circle EFGH, as the cone AL to any ſolid greater
an the cone EN: For, if it be poſſible, let it be ſo to the ſolid
which is greater than the cone EN: Therefore, by inverſion,
the circle EFGH to the circle ABCD, ſo is the ſolid I to
e cone AL: But as the ſolid I to the cone AL, ſo is the
ne EN to ſome ſolid, which muſt be leſs [a] than the cone [a] 14. 5.
L, becauſe the ſolid I is greater than the cone EN: There-
re, as the circle EFGH is to the circle ABCD, ſo is the cone
N to a ſolid leſs than the cone AL, which was ſhewn to be
npoſſible: Therefore the circle ABCD is not to the circle
FGH, as the cone AL is to any ſolid greater than the cone
N: And it has been demonſtrated that neither is the circle
BCD to the circle EFGH, as the cone AL to any ſolid leſs
an the cone EN: Therefore the circle ABCD is to the circle
FGH, as the cone AL to the cone EN: But as the cone is to
e cone, ſo [b] is the cylinder to the cylinder; becauſe the cy- [b] 15. 5.
nders are triple [c] of the cones, each of each. Therefore, as [c] 10. 12.
e circle ABCD to the circle EFGH, ſo are the cylinders
on them of the ſame altitude. Wherefore cones and cylin-
rs of the ſame altitude are to one another as their baſes.
. E. D.

PROP. XII. THEOR.

SIMILAR cones and cylinders have to one another See N.
the triplicate ratio of that which the diameters
their baſes have.

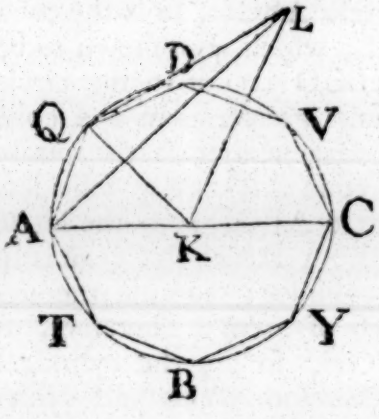
Let the cones and cylinders of which the baſes are the circles
BCD, EFGH, and the diameters of the baſes AC, EG,
d KL, MN the axes of the cones or cylinders, be ſimilar:
he cone of which the baſe is the circle ABCD, and vertex
e point L, has to the cone of which the baſe is the circle
FGH, and vertex N, the triplicate ratio of that which AC
s to EG.

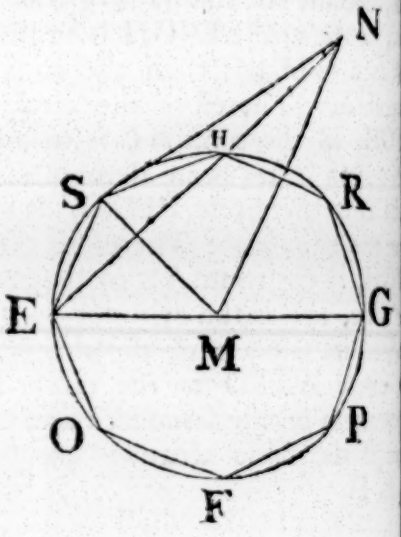
For, if the cone ABCDL has not to the cone EFGHN the
plicate ratio of that which AC has to EG, the cone ABCDL
all have the triplicate of that ratio to ſome ſolid which is leſs

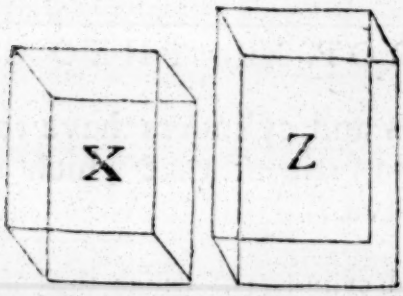
or

Book XII.

or greater than the cone EFGHN. First, let it have it to a le viz. to the ſolid X: Make the ſame conſtruction as in the pre ceeding propoſition, and it may be demonſtrated the very ſam way as in that propoſition, that the pyramid of which the ba is the polygon EOFPGRHS, and vertex N, is greater than th ſolid X Deſcribe alſo in the circle ABCD the polygo ATBYCVDQ ſimilar to the polygon EOFPGRHS, upon whic erect a pyramid having the ſame vertex with the cone; and le LAQ be one of the triangles containing the pyramid upo the polygon ATBYCVDQ the vertex of which is L; and le NES be one of the triangles containing the pyramid upon th







polygon EOFPGRHS of which the vertex is N; and join KQ MS: Becauſe then the cone ABCDL is ſimilar to the con

[a] 24. def. 11. EFGHN, AC is [a] to EG, as the axis KL to the axis MN and as AC to EG, ſo [b] is AK to EM; therefore as AK

[b] 15. 5. EM, ſo is KL to MN; and, alternately, AK to KL, as E to MN: And the right angles AKL, EMN are equal; there fore, the ſides about theſe equal angles being proportional

[c] 6. 6. the triangle AKL is ſimilar [c] to the triangle EMN. Again, be cauſe AK is to KQ, as EM to MS, and that theſe ſides a

abo

bout equal angles AKQ, EMS, becauſe theſe angles are, ach of them, the ſame part of four right angles at the cen- ers K, M; therefore the triangle AKQ is ſimilar [a] to the tri- ngle EMS: And becauſe it has been ſhewn that as AK to KL, is EM to MN, and that AK is equal to KQ, and EM to IS, as QK to KL, ſo is SM to MN; and therefore the ſides bout the right angles QKL, SMN being proportionals, the iangle LKQ is ſimilar to the triangle NMS: And becauſe of e ſimilarity of the triangles AKL, EMN, as LA is to AK, is NE to EM; and by the ſimilarity of the triangles AKQ, MS, as KA to AQ, ſo ME to ES; ex aequali [b], LA is AQ, as NE to ES. Again, becauſe of the ſimilarity of the iangles LQK, NSM, as LQ to QK, ſo NS to SM; and om the ſimilarity of the triangles KAQ, MES, as KQ to A, ſo MS to SE; ex aequali [b], LQ is to QA, as NS to E: And it was proved that QA is to AL, as SE to EN; there- re, again, ex aequali, as QL to LA, ſo is SN to NE: Where- re the triangles LQA, NSE, having the ſides about all their ngles proportionals, are equiangular [c] and ſimilar to one an- ther: And therefore the pyramid of which the baſe is the tri- ngle AKQ, and vertex L, is ſimilar to the pyramid the baſe which is the triangle EMS, and vertex N, becauſe their lid angles are equal [d] to one another, and they are contained the ſame number of ſimilar planes: But ſimilar pyramids hich have triangular baſes have to one another the triplicate ratio of that which their homologous ſides have; therefore e pyramid AKQL has to the pyramid EMSN the triplicate tio of that which AK has to EM In the ſame manner, if raight lines be drawn from the points D, V, C, Y, B, T K, and from the points H, R, G, P, F, O to M, and py- mids be erected upon the triangles having the ſame vertices ith the cones, it may be demonſtrated that each pyramid in e firſt cone has to each in the other, taking them in the ſame der, the triplicate ratio of that which the ſide AK has to the de EM; that is, which AC has to EG: But as one antece- ent to its conſequent, ſo are all the antecedents to all the nſequents [f]; therefore as the pyramid AKQL to the pyra- id EMSN, ſo is the whole pyramid the baſe of which is the olygon DQATBYCV, and vertex L, to the whole pyramid which the baſe is the polygon HSE FPGR, and vertex N. herefore alſo the firſt of theſe two laſt named pyramids has the other the triplicate ratio of that which AC has to EG.

[a] 6. 6.

[b] 22. 5.

[c] 5. 6.

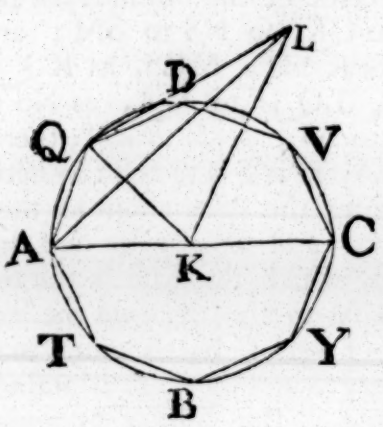
[d] B. 11.

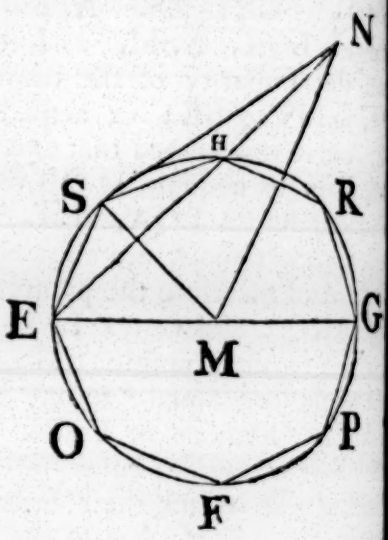
[e] 8. 12.

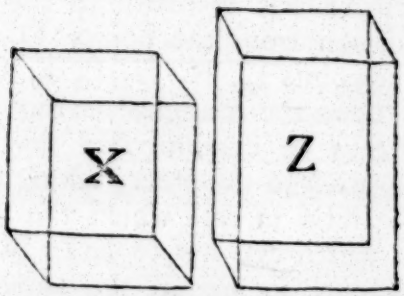
[f] 12. 5.

But,

Book XII But, by the hypotheſis, the cone of which the baſe is the cir-
cle ABCD, and vertex L has to the ſolid X, the triplicate
ratio of that which AC has to EG; therefore, as the cone
which the baſe is the circle ABCD, and vertex L, is to the
ſolid X, ſo is the pyramid the baſe of which is the polygon
DQATBYCV, and vertex L, to the pyramid the baſe of which
is the polygon HSEOFPGR and vertex N: But the ſaid cone
is greater than the pyramid contained in it, therefore the ſolid
[a] 14. 5. X is greater [a] than the pyramid the baſe of which is the polygon HSEOFPGR, and vertex N; but it is alſo leſs; which is
impoſſible: Therefore the cone of which the baſe is the circle







ABCD, and vertex L, has not to any ſolid which is leſs than the
cone of which the baſe is the circle EFGH and vertex N, the tri-
plicate ratio of that which AC has to EG. In the ſame manner
it may be demonſtrated that neither has the cone EFGHN to
any ſolid which is leſs than the cone ABCDL, the triplicate
ratio of that which EG has to AC. Nor can the cone ABCDL
have to any ſolid which is greater than the cone EFGHN, the
triplicate ratio of that which AC has to EG: For, if it be poſ-
ſible, let it have it to a greater, viz. to the ſolid Z: Therefore,
inverſ

verſely, the ſolid Z has to the cone ABCDL the triplicate tio of that which EG has to AC: But as the ſolid Z is to e cone ABCDL, ſo is the cone EFGHN to ſome ſolid, hich muſt be leſs [a] than the cone ABCDL, becauſe the ſolid is greater than the cone EFGHN: Therefore the cone EFGHN s to a ſolid which is leſs than the cone ABCDL, the tripli- te ratio of that which EG has to AC, which was demon- ated to be impoſſible: Therefore the cone ABCDL has not any ſolid greater than the cone EFGHN, the triplicate ra- of that which AC has to EG; and it was demonſtrated at it could not have that ratio to any ſolid leſs than the cone FGHN: Therefore the cone ABCDL has to the cone EFGHN, e triplicate ratio of that which AC has to EG: But as the ne is to the cone, ſo [b] the cylinder to the cylinder; for eve- cone is the third part of the cylinder upon the ſame baſe, d of the ſame altitude: Therefore alſo the cylinder has to the linder, the triplicate ratio of that which AC has to EG. Therefore ſimilar cones, &c. Q. E. D.

[a] 14. 5.

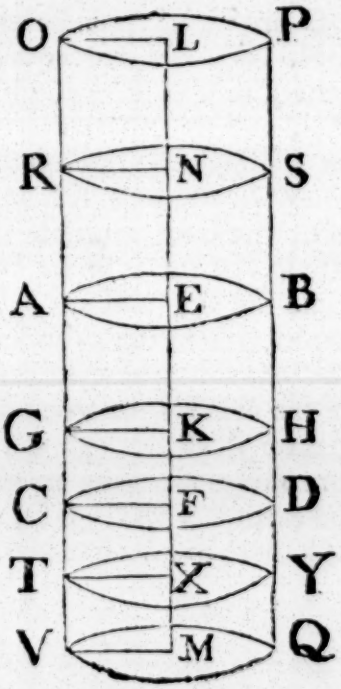
[b] 15. 5.

PROP. XIII. THEOR.

F a cylinder be cut by a plane parallel to its oppo- ſite planes, or baſes; it divides the cylinder into wo cylinders, one of which is to the other as the xis of the firſt to the axis of the other.

See N.

Let the cylinder AD be cut by the ane GH parallel to the oppoſite anes AB, CD, meeting the axis in the point K, and let the line H be the common ſection of the ane GH and the ſurface of the cy- der AD: Let AEFC be the paral- ogram, in any poſition of it, by the olution of which about the ſtraight e EF the cylinder AD is deſcribed; d let GK be the common ſection the plane GH, and the plane EFC: And becauſe the parallel nes AB, GH are cut by the plane EKG, AE, KG, their common tions with it, are parallel [a]; where- e AK is a parallelogram, and GK ual to EA the ſtraight line from center of the circle AB: For the ne reaſon, each of the ſtraight lines



[a] 16. 11.

drawn

Book XII. drawn from the point K to the line GH may be proved to be equal to thoſe which are drawn from the center of the circle AB to its circumference, and are therefore all equal to one another. Therefore the line GH is the circumference of a circle of which the center is the point K: Therefore the plane GH divides the cylinder AD into the cylinders AH, GD; for they are the ſame which would be deſcribed by the revolution of the parallelograms AK, GF about the ſtraight lines EK, KF: And it is to be ſhewn that the cylinder AH is to the cylinder HC as the axis EK to the axis KF.

[a] 15. def. 1.

Produce the axis EF both ways; and take any number of ſtraight lines EN, NL, each equal to EK; and any number FX, XM, each equal to FK; and let planes parallel to AB, CD paſs thro' the points L, N, X, M: Therefore the common ſections of theſe planes with the cylinder produced are circles the centers of which are the points L, N, X, M, as was proved of the plane GH; and theſe planes cut off the cylinders PR, RB, DT, TQ: And becauſe the axes LN, NE, EK are all equal; therefore the cylinders PR, RB, BG are [b] to one another as their baſes; But their baſes are equal, and therefore the cylinders PR, RB, BG are equal: And becauſe the axes LN, NE, EK are equal to one another, as alſo the cylinders PR, RB, BG, and that there are as many axes as cylinders; therefore, whatever multiple the axis KL is of the axis KE, the ſame multiple is the cylinder PG of the cylinder GB: For the ſame reaſon, whatever multiple the axis MK is of the axis KF, the ſame multiple is the cylinder QG of the cylinder CD. And if the axis KL be equal to the axis KM, the cylinder PG is equal to the cylinder GQ; and if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if leſs, leſs: Since therefore there are four magnitudes, viz. the axes EK, KF, and the cylinders BG, GD, and that of the axis EK and cylinder BG there has been taken any equimultiples whatever, viz.

[b] 11. 12.

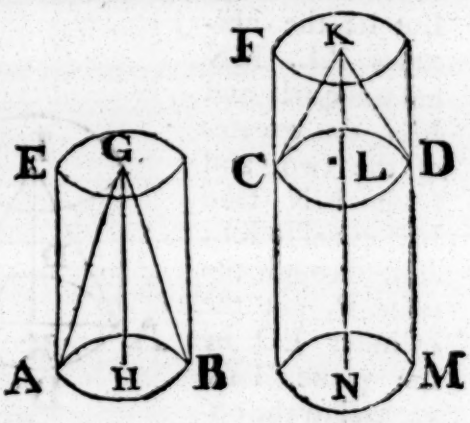
is KL and cylinder PG; and of the axis KF and cylinder D, any equimultiples whatever, viz. the axis KM and cylinder GQ; and it has been demonſtrated, if the axis KL be eater than the axis KM, the cylinder PG is greater than the linder GQ; and if equal, equal; and if leſs, leſs: Therefore the axis EK is to the axis KF, as the cylinder BG to the cylinder GD. Wherefore, if a cylinder, &c. Q. E. D. Book XII. [d] 5. def. 5.

PROP. XIV. THEOR.

CONES and cylinders upon equal baſes are to one another as their altitudes.

Let the cylinders EB, FD be upon the equal baſes AB, CD: s the cylinder EB to the cylinder FD, ſo is the axis GH to e axis KL.

Produce the axis KL to the point N, and make LN equal the axis GH, and let CM be a cylinder of which the baſe is D, and axis LN: and becauſe the cylinders EB, CM have e ſame altitude, they are to one another as their baſes [a]: But eir baſes are equal, therefore alſo the cylinders EB, CM are ual. And becauſe the cylinder FM is cut by the plane D parallel to its oppoſite anes, as the cylinder CM to e cylinder FD, ſo is [b] the is LN to the axis KL. But e cylinder CM is equal to e cylinder EB, and the axis N to the axis GH: Therefore as the cylinder EB to the linder FD, ſo is the axis H to the axis KL: And as e cylinder EB to the cylinder FD, ſo is [c] the cone ABG to the ne CDK, becauſe the cylinders are triple [d] of the cones: herefore alſo the axis GH is to the axis KL, as the cone ABG the cone CDK, and the cylinder EB to the cylinder FD. herefore cones, &c. Q. E. D.



[a] 11. 12. [b] 13. 12. [c] 15. 5. [d] 10. 12.

Book XII.

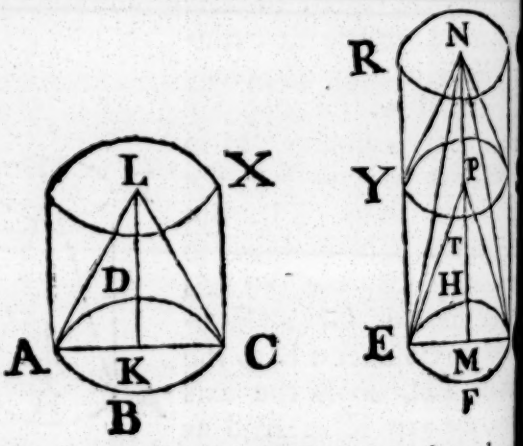
PROP. XV. THEOR.

See N. THE baſes and altitudes of equal cones and cylinders, are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the circles ABCD, EFGH, the diameters of which are AC, EG, be the baſes, and KL, MN the axes, as alſo the altitudes, of equal cones and cylinders; and let ALC, EON be the cones, and AX, EO the cylinders: The baſes and altitudes of the cylinders AX, EO are reciprocally proportional, that is, as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL.

Either the altitude MN is equal to the altitude KL, or the altitudes are not equal. Firſt, let them be equal; and the cylinders AX, EO being alſo equal, and cones and cylinders of the ſame altitude being to one another as their baſes [a], therefore the baſe ABCD is equal [b] to the baſe EFGH; and as the baſe ABCD is to the baſe EFGH, ſo is the altitude MN to the altitude KL.

[a] 11. 12
[b] A. 5.



But let the altitudes KL, MN be unequal, and MN the greater of the two, and from MN take MP equal to KL, and, through the point P, cut the cylinder EO by the plane TYS parallel to the oppoſite planes of the circles EFGH, RO; therefore the common ſection of the plane TYS and the cylinder EO is a circle, and conſequently ES is a cylinder, the baſe of which is the circle EFGH, and altitude MP: And becauſe the cylinder AX is equal to the cylinder EO, as AX is to the cylinder ES [c] is the cylinder EO to the ſame ES. But as the cylinder EO to the cylinder ES, ſo [a] is the baſe ABCD to the baſe EFGH, for the cylinders AX, ES are of the ſame altitude; and as the cylinder EO to the cylinder ES, ſo [d] is the altitude MN to the altitude MP, becauſe the cylinder EO is cut by the

[c] 7. 5.
[d] 13. 12.

ſs parallel to its oppoſite planes. Therefore as the baſe Book XII.
BCD to the baſe EFGH, ſo is the altitude MN to the altitude
P: But MP is equal to the altitude KL; wherefore as the
e ABCD to the EFGH, ſo is the altitude MN to the altitude
L; that is, the baſes and altitudes of the equal cylinders AX,
are reciprocally proportional.

But let the baſes and altitudes of the cylinders AX, EO, be
iprocally proportional, viz. the baſe ABCD to the baſe
GH, as the altitude MN to the altitude KL: The cylinder
X is equal to the cylinder EO.

Firſt, let the baſe ABCD be equal to the baſe EFGH; then,
cauſe as the baſe ABCD is to the baſe EFGH, ſo is the al-
de MN to the altitude KL; MN is equal [b] to KL, and [b] A. 5.
refore the cylinder AX is equal [a] to the cylinder EO. [a] 11. 12.

But let the baſes ABCD, EFGH be unequal, and let ABCD
he greater; and becauſe, as ABCD is to the baſe EFGH, ſo
he altitude MN to the altitude KL; therefore MN is great-
than KL. Then, the ſame conſtruction being made as be-
e, becauſe as the baſe ABCD to the baſe EFGH, ſo is the al-
de MN to the altitude KL; and becauſe the altitude KL
qual to the altitude MP; therefore the baſe ABCD is [a] to
baſe EFGH, as the cylinder AX to the cylinder ES; and
he altitude MN to the altitude MP or KL, ſo is the cylinder
to the cylinder ES: Therefore the cylinder AX is to the
inder ES, as the cylinder EO is to the ſame ES: Whence the
inder AX is equal to the cylinder EO: And the ſame reaſon-
holds in cones. Q. E. D.

PROP. XVI. PROB.

TO deſcribe in the greater of two circles that have the ſame center, a polygon of an even number equal ſides, that ſhall not meet the leſſer circle.

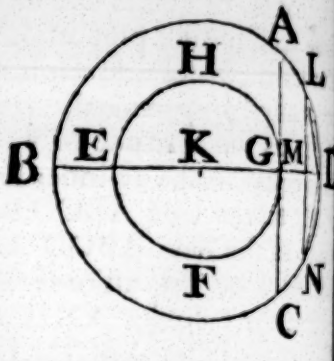
Let ABCD, EFGH be two given circles having the ſame cen-
K: It is required to inſcribe in the greater circle ABCD
olygon of an even number of equal ſides, that ſhall not meet
leſſer circle.

Through the center K draw the ſtraight line BD, and from
point G, where it meets the circumference of the leſſer

Book XII. circle, draw GA at right angles to BD, and produce it to C; therefore AC touches [a] the circle EFGH: Then, if the circumference BAD be biſected, and the half of it be again biſected, and ſo on, there muſt at length remain a circumference leſs than AD: Let this be LD; and from the point L draw LM perpendicular to BD, and produce it to N; and join LD, DN. Therefore LD is equal to DN; and becauſe LN is parallel to AC, and that AC touches the circle EFGH; therefore LN does not meet the circle EFGH: And much leſs ſhall the ſtraight lines LD, DN meet the circle EFGH: ſo that if ſtraight lines equal to LD be applied in the circle ABCD from the point L around to N, there ſhall be deſcribed in the circle a polygon of an even number of equal ſides not meeting the leſſer circle. Which was to be done.

[a] 16. 3

[b] Lemma.



LEMMA II.

IF two trapeziums ABCD, EFGH be inſcribed in the circles, the centers of which are the points K, L; and if the ſides AB, DC be parallel, as alſo EF, HG; and the other four ſides AD, BC, EH, FG be all equal to one another; but the ſide AB greater than EF, and DC greater than HG. The ſtraight line KA from the center of the circle in which the greater ſides are, is greater than the ſtraight line LE drawn from the center to the circumference of the other circle.

If it be poſſible, let KA be not greater than LE; then it muſt be either equal to it, or leſs. Firſt, let KA be equal to LE: Therefore, becauſe in two equal circles, AD, BC in one are equal to EH, FG in the other, the circumferences AD, BC are equal [a] to the circumferences EH, FG; becauſe the ſtraight lines AB, DC are reſpectively greater than EF, GH, the circumferences AB, DC are greater than EH, HG: Therefore the whole circumference ABCD is greater than the whole EFGH; but it is alſo equal to it, which is impoſſible.

[a] 28. 3.

poſſible: Therefore the ſtraight line KA is not equal to Book XII.
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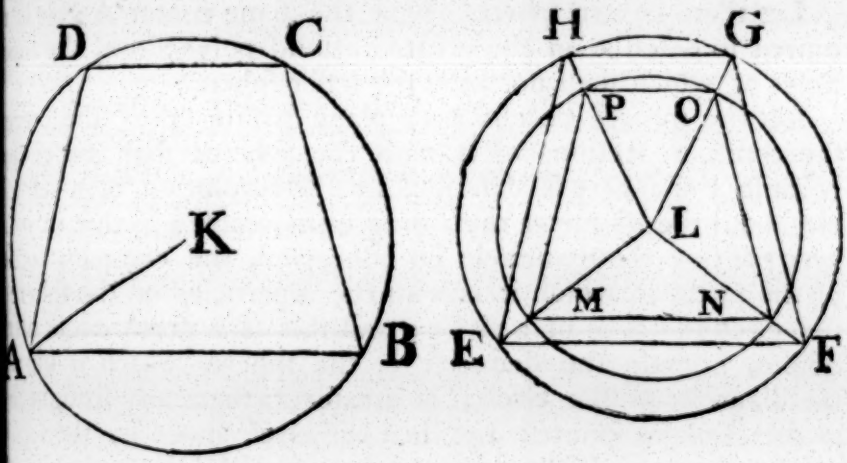
But let KA be leſs than LE, and make LM, equal to KA, d from the center L, and diſtance LM deſcribe the circle NOP, meeting the ſtraight lines LE, LF, LG, LH, in M, O, P; and join MN, NO, OP, PM which are reſpective- parallel [a] to, and leſs than EF, FG, GH, HE: Then, be- uſe EH is greater than MP, AD is greater than MP; and [a] 2. 6.



e circles ABCD, MNOP are equal; therefore the circum- rence AD is greater than MP; for the ſame reaſon, the cir- mference BC is greater than NO; and becauſe the ſtraight e AB is greater than EF which is greater than MN, much ore is AB greater than MN: Therefore the circumference B is greater than MN; and for the ſame reaſon, the circum- rence DC is greater than PO: Therefore the whole circumfe- nce ABCD is greater than the whole MNOP; but it is like- iſe equal to it, which is impoſſible: Therefore KA is not leſs an LE; nor is it equal to it; the ſtraight line KA muſt erefore be greater than LE. Q. E. D.

Cor. And if there be an iſoſceles triangle the ſides of which e equal to AD, BC, but its baſe leſs than AB the greater of e two ſides AB, DC; the ſtraight line KA may, in the ſame anner, be demonſtrated to be greater than the ſtraight line rawn from the center to the circumference of the circle de- ribed about the triangle.

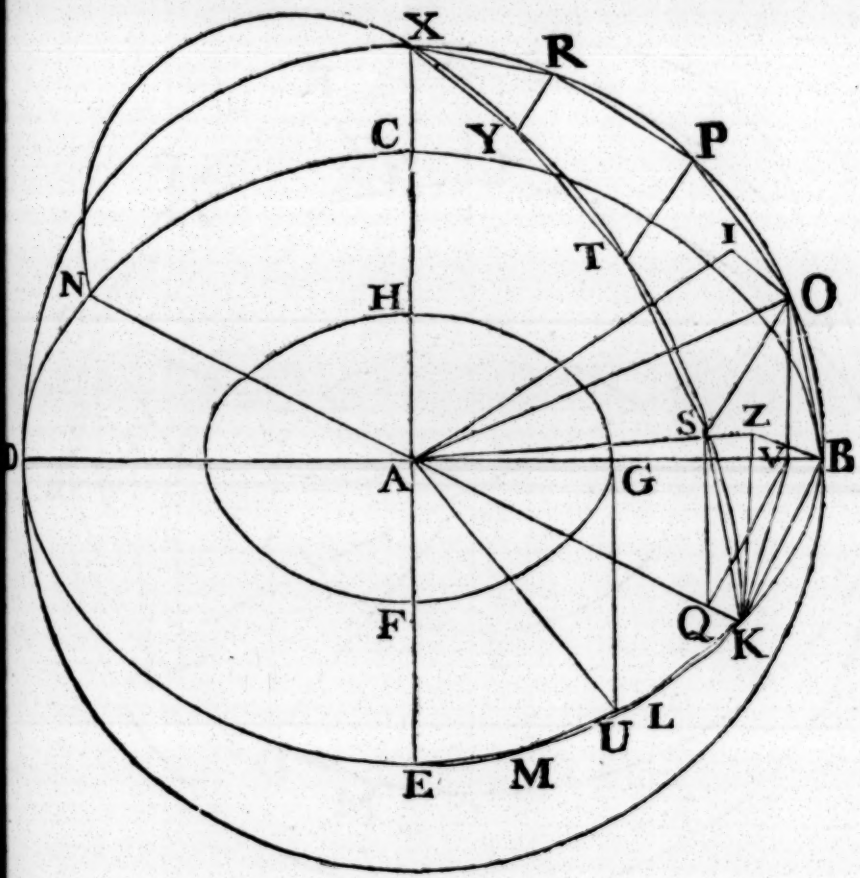
PROP. XVII. PROB.

See N. TO deſcribe in the greater of two ſpheres which have the ſame center, a ſolid polyhedron, the ſuperficies of which ſhall not meet the leſſer ſphere.

Let there be two ſpheres about the ſame center A; it is required to deſcribe in the greater a ſolid polyhedron, the ſuperficies of which ſhall not meet the leſſer ſphere.

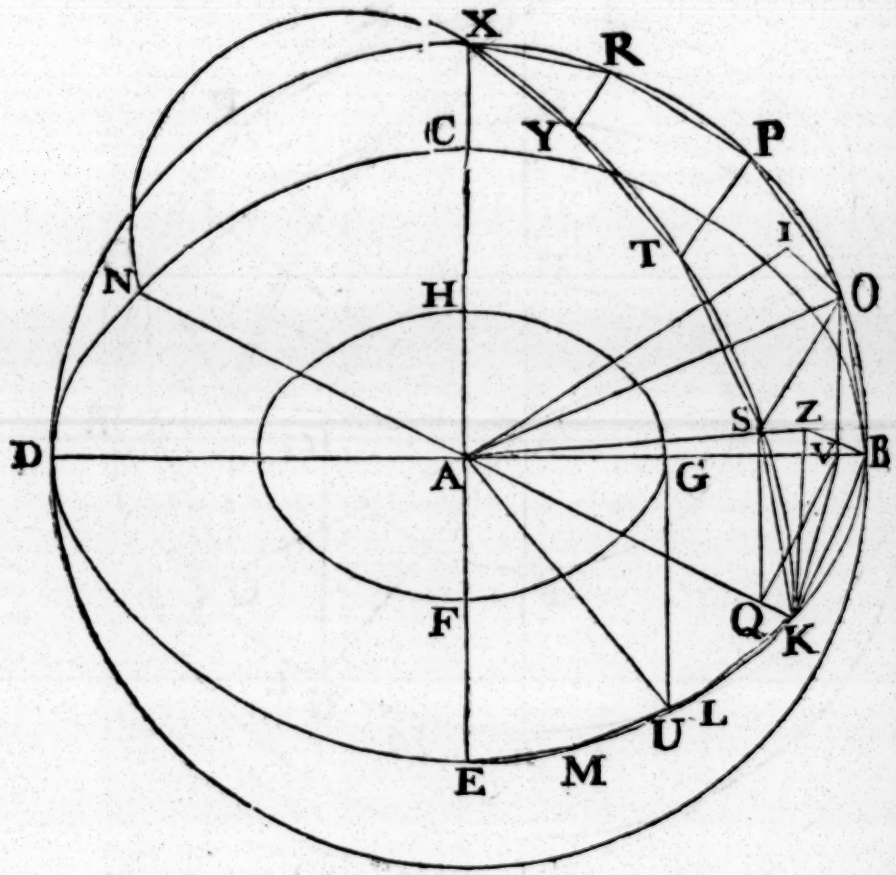
Let the ſpheres be cut by a plane paſſing thro' the center, the common ſections of it with the ſpheres ſhall be circles, becauſe the ſphere is deſcribed by the revolution of a ſemicircle about the diameter remaining unmoveable; ſo that in whatever poſition the ſemicircle be conceived, the common ſection of the plane in which it is with the ſuperficies of the ſphere is the circumference of a circle; and this is a great circle of the ſphere, becauſe the diameter of the ſphere, which is likewiſe [a] 15. 3. the diameter of the circle, is greater [a] than any ſtraight line in the circle or ſphere: Let then the circle made by the ſection of the plane with the greater ſphere be BCDE, and with the leſſer ſphere be FGH; and draw the two diameters BD, CE at right angles to one another: And in BCDE, the greater of [b] 16. 12. the two circles, deſcribe [b] a polygon of an even number of equal ſides not meeting the leſſer circle FGH; and let its ſides in BE, the fourth part of the circle, be BK, KL, LM, ME; join KA and produce it to N; and from A draw AX at right angles to the plane of the circle BCDE meeting the ſuperficies of the ſphere in the point X; and let planes paſs thro' AX and each of the ſtraight lines BD, KN, which, from what has been ſaid, ſhall produce great circles on the ſuperficies of the ſphere, and let BXD, KXN be the ſemicircles thus made upon the diameters BD, KN: Therefore, becauſe XA is at right angles to the plane of the circle BCDE, every plane which [c] 18. 11. paſſes thro' XA is at right [c] angles to the plane of the circle BCDE; wherefore the ſemicircles BXD, KXN are at right angles to that plane: And becauſe the ſemicircles BED, BXD, KXN, upon the equal diameters BD, KN, are equal to one another, their halves BE, BX, KX, are equal to one another: Therefore, as many ſides of the polygon as are in BE ſo many there are in BX, KX equal to the ſides BK, KL, LM, ME: Let theſe polygons be deſcribed, and the ſides be BO, OP, PR, RX; KS, ST, TY, YX, and join

, PT, RY; and from the points O, S draw OV, SQ perpen- Book XII.
ulars to AB, AK: And becauſe the plane BOXD is at right
les to the plane BCDE, and in one of them BOXD, OV
rawn perpendicular to AB the common ſection of the planes,
refore OV is perpendicular [a] to the plane BCDE: For the [a] 4. def.
e reaſon SQ is perpendicular to the ſame plane, becauſe 11.
plane KSXN is at right angles to the plane BCDE. Join
Q, and becauſe in the equal ſemicircles BXD, KXN the



rcumferences BO, KS are equal, and OV, SQ are perpen-
cular to their diameters, therefore [d] OV is equal to SQ, [d] 26. 1.
d BV equal to KQ: But the whole BA is equal to the whole
A, therefore the remainder VA is equal to the remainder
A: As therefore BV is to VA, ſo is KQ to QA, wherefore
Q is parallel [e] to BK: And becauſe OV, SQ are each of [e] 2. 6.
em at right angles to the plane of the circle BCDE, OV is
rallel [f] to SQ; and it has been proved that it is alſo equal [f] 6 11.
it; therefore OV, SO are equal and parallel [g]: And becauſe [g] 33. 1.
V is parallel to SO, and alſo to KB; OS is parallel [h] to BK; [h] 9. 11.
d therefore BO, KS which join them are in the ſame plane

Book XII. in which theſe parallels are, and the quadrilateral figure KBO is in one plane: And if PB, TK be joined, and perpendiculars be drawn from the points P, T to the ſtraight lines AB, AK it may be demonſtrated that TP is parallel to KB in the ver ſame way that SO was ſhewn to be parallel to the ſame KB
[a] 9. 11. wherefore [a] TB is parallel to SO, and the quadrilateral figur SOPT is in one plane: For the ſame reaſon, the quadrilatera
[b] 2. 11. TPRY is in one plane: And the figure YRX is alſo in one plane[b]



Therefore, if from the points, O, S, P, T, R, Y there be drawn ſtraight lines to the point A, there ſhall be formed a ſolid po lyhedron between the circumferences BX, KX compoſed o pyramids the baſes of which are the quadrilaterals KBOS SOPT, TPRY, and the tringle YRX, and of which the com mon vertex is the point A: And if the ſame conſtruction b made upon each of the ſides KL, LM, ME, as has been don upon BK, and the like be done alſo in the other three qua drants, and in the other hemiſphere; there ſhall be for med a ſolid polyhedron deſcribed in the ſphere, compo ſe

ed of pyramids the baſes of which are the aforeſaid quadrilateral figures, and the triangle YRX, and thoſe formed in the like manner in the reſt of the ſphere, the common vertex of them all being the point A: And the ſuperficies of this ſolid polyhedron does not meet the leſſer ſphere in which is the circle FGH; For, from the point A draw [a] AZ perpendicular to the plane of the quadrilateral KBOS meeting it in Z, and join BZ, ZK: And becauſe AZ is perpendicular to the plane KBOS, it makes right angles with every ſtraight line meeting it in that plane; therefore AZ is perpendicular to BZ and ZK: And becauſe AB is equal to AK, and that the ſquares of AZ, ZB, are equal to the ſquare of AB; and the ſquares of AZ, ZK to the ſquare of AK [b]; therefore the ſquares of AZ, ZB are equal to the ſquares of AZ, ZK: Take from theſe equals the ſquare of AZ, the remaining ſquare of BZ is equal to the remaining ſquare of ZK; and therefore the ſtraight line BZ is equal to ZK: In the like manner it may be demonſtrated, that the ſtraight lines drawn from the point Z to the points O, S are equal to BZ or ZK: Therefore the circle deſcribed from the center Z, and diſtance ZB ſhall paſs through the points K, O, S, and KBOS ſhall be a quadrilateral figure in the circle: And becauſe KB is greater than QV, and QV equal to SO, therefore KB is greater than SO: But KB is equal to each of the ſtraight lines BO, KS; wherefore each of the circumferences cut off by KB, BO, KS is greater than that cut off by OS; and theſe three circumferences together with a fourth equal to one of them, are greater than the ſame three together with that cut off by OS; that is, than the whole circumference of the circle; therefore the circumference ſubtended by KB is greater than the fourth part of the whole circumference of the circle KBOS, and conſequently the angle BZK at the center is greater than a right angle: And becauſe the angle BZK is obtuſe, the ſquare of BK is greater [c] than the ſquares of BZ, ZK; that is, greater than twice the ſquare of BZ. Join KV, and becauſe in the triangles KBV, OBV, KB, BV are equal to OB, BV, and that they contain equal angles; the angle KVB is equal [d] to the angle OVB: And OVB is a right angle; therefore alſo KVB is a right angle; And becauſe BD is leſs than twice DV, the rectangle contained by DB, BV is leſs than twice the rectangle DVB; that is [e], the ſquare of KB is leſs than twice the ſquare of KV: But the ſquare of KB is greater than twice the ſquare of BZ; therefore the ſquare of KV is greater

[a] 11. 11.

[b] 47. 1.

[c] 12. 2.

[d] 4. 1.

[e] 8. 6.

Book. XII.

greater than the ſquare of BZ: And becauſe BA is equal to AK, and that the ſquares of BZ, ZA are equal together to the ſquare of BA, and the ſquares of KV, VA to the ſquare of AK; therefore the ſquares of BZ, ZA are equal to the ſquares of KV, VA; and of theſe the ſquare of KV is greater than the ſquare of BZ; therefore the ſquare of VA is leſs than the ſquare of ZA, and the ſtraight line AZ greater than VA: Much more then is AZ greater than AG; becauſe, in the preceeding propoſition, it was ſhewn that KV falls without the circle FGH: And AZ is perpendicular to the plane KBOS, and is therefore the ſhorteſt of all the ſtraight lines that can be drawn from A the center of the ſphere to that plane. Therefore the plane KBOS does not meet the leſſer ſphere.

And that the other planes between the quadrants BX, KX fall without the leſſer ſphere, is thus demonſtrated: From the point A draw AI perpendicular to the plane of the quadrilateral SOPT, and join IO; and, as was demonſtrated of the plane KBOS and the point Z, in the ſame way it may be ſhewn that the point I is the center of a circle deſcribed about SOPT; and that OS is greater than PT; and PT was ſhewn to be parallel to OS: Therefore, becauſe the two trapeziums KBOS, SOPT inſcribed in circles have their ſides BK, OS parallel, and alſo OS, PT; and their other ſides BO, KS, OP, ST all equal to one another, and that BK is greater than OS, and OS
*2. Lem. 12. greater than PT, therefore the ſtraight line ZB is greater than IO. Join AO which will be equal to AB; and becauſe AIO, AZB are right angles, the ſquares of AI, IO are equal to the ſquare of AO or of AB; that is, to the ſquares of AZ, ZB; and the ſquare of ZB is greater than the ſquare of IO; therefore the ſquare of AZ is leſs than the ſquare of AI; and the ſtraight line AZ leſs than the ſtraight line AI: And it was proved that AZ is greater than AG; much more then is AI greater than AG: Therefore the plane SOPT falls wholly without the leſſer ſphere: In the ſame manner it may be demonſtrated that the plane TPRY falls without the ſame ſphere as alſo the triangle YRX, viz. by the Cor. of 2d Lemma. And after the ſame way it may be demonſtrated that all the planes which contain the ſolid polyhedron fall without the leſſer ſphere. Therefore in the greater of two ſpheres which have the ſame center, a ſolid polyhedron is deſcribed, the ſuperficies of which does not meet the leſſer ſphere. Which was to be done.

But the ſtraight line AZ may be demonſtrated to be greater than AG otherwiſe and in a ſhorter manner, without the help of Prop. 16. as follows. From the point G draw GU at right angles to AG and join AU. If then the circumference BE be biſected, and its half again biſected, and ſo on, there will at length be left a circumference leſs than the circumference which is ſubtended by a ſtraight line equal to GU inſcribed in the circle BCDE: Let this be the circumference KB: Therefore the ſtraight line KB is leſs than GU: And becauſe the angle BZK is obtuſe, as was proved in the preceeding, therefore BK is greater than BZ: But GU is greater than BK; much more then is GU greater than BZ, and the ſquare of GU than the ſquare of BZ: And AU is equal to AB; therefore the ſquare of AU, that is, the ſquares of AG, GU are equal to the ſquare of AB, that is, to the ſquares of AZ, ZB; but the ſquare of BZ is leſs than the ſquare of GU; therefore the ſquare of AZ is greater than the ſquare of AG, and the ſtraight line AZ conſequently greater than the ſtraight line AG.

Cor. And if in the leſſer ſphere there be deſcribed a ſolid polyhedron by drawing ſtraight lines betwixt the points in which the ſtraight lines from the center of the ſphere drawn to all the angles of the ſolid polyhedron in the greater ſphere meet the ſuperficies of the leſſer; in the ſame order in which are joined the points in which the ſame lines from the center meet the ſuperficies of the greater ſphere; the ſolid polyhedron in the ſphere BCDE has to this other ſolid polyhedron the triplicate ratio of that which the diameter of the ſphere BCDE has to the diameter of the other ſphere: For, if theſe two ſolids be divided into the ſame number of pyramids, and in the ſame order; the pyramids ſhall be ſimilar to one another, each to each: Becauſe they have the ſolid angles at their common vertex, the center of the ſphere, the ſame in each pyramid, and their other ſolid angles at the baſes equal to one another, each to each [a], becauſe they are contained by three plane angles equal each to each; and the pyramids are contained by the ſame number of ſimilar planes; and are therefore ſimilar [b] to one another, each to each: But ſimilar pyramids have to one another the triplicate [c] ratio of their homologous ſides. Therefore the pyramid of which the baſe is the quadrilateral KBOS, and vertex A, has to the pyramid in the other ſphere of the ſame order, the triplicate ratio of their homologous ſides;

[a] B. 11.

[b] 11. def. 11.

[c] Cor. 12.

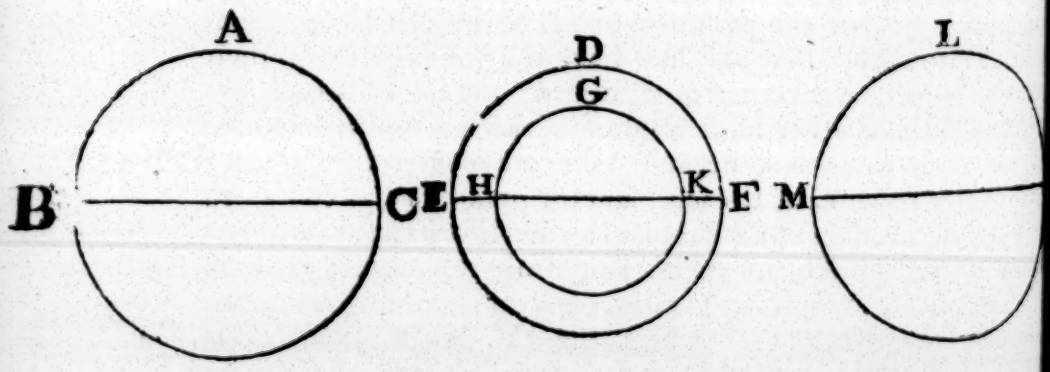
Book XII. ſides; that is, of that ratio which AB from the center of the greater ſphere has to the ſtraight line from the ſame center to the ſuperficies of the leſſer ſphere. And in like manner, each pyramid in the greater ſphere has to each of the ſame order in the leſſer, the triplicate ratio of that which AB has to the ſemidiameter of the leſſer ſphere. And as one antecedent is to its conſequent, ſo are all the antecedents to all the conſequents. Wherefore the whole ſolid polyhedron in the greater ſphere has to the whole ſolid polyhedron in the other, the triplicate ratio of that which AB the ſemidiameter of the firſt has to the ſemidiameter of the other; that is, which the diameter BD of the greater has to the diameter of the other ſphere.

PROP. XVIII. THEOR.

SPHERES have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF be two ſpheres of which the diameters are BC, EF. The ſphere ABC has to the ſphere DEF the triplicate ratio of that which BC has to EF.

For, if it has not, the ſphere ABC ſhall have to a ſphere either leſs or greater than DEF, the triplicate ratio of that which BC has to EF. Firſt, let it have that ratio to a leſs, viz. to the ſphere GHK; and let the ſphere DEF have the ſame
[a] 17. 12. center with GHK; and in the greater ſphere DEF deſcribe



a ſolid polyhedron, the ſuperficies of which does not meet the leſſer ſphere GHK; and in the ſphere ABC deſcribe another ſimilar to that in the ſphere DEF: Therefore the ſolid polyhedron in the ſphere ABC has to the ſolid polyhedron in the
[b] Cor. 17. 12. ſphere DEF, the triplicate ratio [b] of that which BC has to EF. But the ſphere ABC has to the ſphere GHK, the triplicate

o of that which BC has to EF; therefore as the ſphere ABC o the ſphere GHK, ſo is the ſolid polyhedron in the ſphere ABC o the ſolid polyhedron in the ſphere DEF: But the ſphere ABC is greater than the ſolid polyhedron in it; therefore [c] al- o the ſphere GHK is greater than the ſolid polyhedron in the phere DEF: But it is alſo leſs, becauſe it is contained within , which is impoſſible: Therefore the ſphere ABC has not to ny ſphere leſs than DEF, the triplicate ratio of that which BC has to EF. In the ſame manner, it may be demonſtrated, hat the ſphere DEF has not to any ſphere leſs than ABC, the riplicate ratio of that which EF has to BC. Nor can the phere ABC have to any ſphere greater than DEF, the tripli- ate ratio of that which BC has to EF: For, if it can, let it ave that ratio to a greater ſphere LMN: Therefore, by inver- ion, the ſphere LMN has to the ſphere ABC, the triplicate atio of that which the diameter EF has to the diameter BC. But, as the ſphere LMN to ABC, ſo is the ſphere DEF to ſome phere, which muſt be leſs [c] than the ſphere ABC, becauſe the phere LMN is greater than the ſphere DEF: Therefore the phere DEF has to a ſphere leſs than ABC the triplicate ratio f that which EF has to BC; which was ſhewn to be impoſ- ible: Therefore the ſphere ABC has not to any ſphere greater han DEF the triplicate ratio of that which BC has to EF: And t was demonſtrated, that neither has it that ratio to any ſphere eſs then DEF. Therefore the ſphere ABC has to the ſphere DEF, the triplicate ratio of that which BC has to EF. Q. E. D.

[c] 14. 5.

FINIS.

16 OC 65

NOTES

CRITICAL AND GEOMETRICAL;

CONTAINING

An Account of thoſe Things in which this Edition differs from the Greek Text; and the Reaſons of the Alterations which have been made. As alſo Obſervations on ſome of the Propoſitions.

BY

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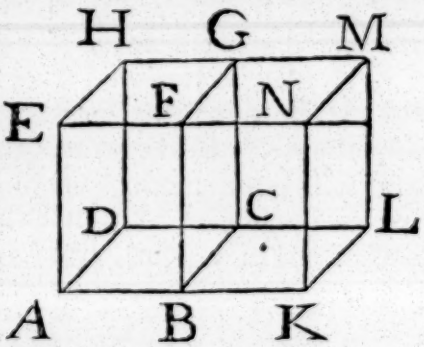
M, DCC, LXXI.

BRITISH MUSEUM

NOTES, &c.

DEFINITION I. BOOK I.

IT is neceſſary to conſider a ſolid, that is, a magnitude which has length, breadth, and thickneſs, in order to underſtand right the definitions of a point, line, and ſuperficies; for theſe all ariſe from a ſolid, and exiſt in it: The boundary, or boundaries which contain a ſolid are called ſuperficies, or the boundary which is common to two ſolids which are contiguous, or which divides one ſolid into two contiguous parts, is called a ſuperficies: Thus, if BCGF be one of the boundaries which contain the ſolid ABCDEFGH, or which is the common boundary of this ſolid, and the ſolid BKLCENMG, and is therefore in the one as well as the other ſolid, is called a ſuperficies, and has no thickneſs: For if it have any, this thickneſs muſt either be a part of the thickneſs of the ſolid AG, or the ſolid BM, or a part of the thickneſs of each of them. It cannot be a part of the thickneſs of the ſolid BM; becauſe, if this ſolid be removed from the ſolid AG, the ſuperficies BCGF, the boundary of the ſolid AG, remains ſtill the ſame as it was. Nor can it be a part of the thickneſs of the ſolid AG; becauſe, if this be removed from the ſolid BM, the ſuperficies BCGF, the boundary of the ſolid BM, does nevertheleſs remain; therefore the ſuperficies BCGF has no thickneſs, but only length and breadth.

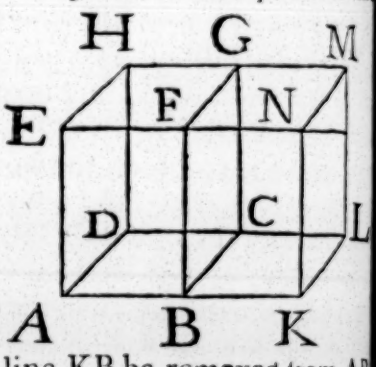


The boundary of a ſuperficies is called a line, or a line is the common boundary of two ſuperficies that are contiguous, or which divides one ſuperficies into two contiguous parts: Thus, if BC be one of the boundaries which contain the ſuperficies ABCD, or which is the common boundary of this ſuperficies, and of the ſuperficies KBCL which is contiguous to it, this boundary BC is called a line, and has no breadth: For, if it have any, this muſt be part either of the breadth of the ſuperficies ABCD, or of the ſuperficies KBCL, or part of each of them. It is not part of the breadth of the ſuperficies KBCL; for, if this ſuperficies be removed from the ſuperficies ABCD,

Book I.

the line BC which is the boundary of the ſuperficies ABCD remains the ſame as it was: Nor can the breadth that BC is ſuppoſed to have, be a part of the breadth of the ſuperficies ABCD, becauſe, if this be removed from the ſuperficies KBCL, the line BC which is the boundary of the ſuperficies KBCL does nevertheleſs remain: Therefore the line BC has no breadth: And becauſe the line BC is in a ſuperficies, and that a ſuperficies has no thickneſs, as was ſhewn; therefore a line has neither breadth nor thickneſs, but only length.

The boundary of a line is called a point, or a point is the common boundary or extremity of two lines that are contiguous: Thus, if B be the extremity of the line AB, or the common extremity of the two lines AB, KB, this extremity is called a point, and has no length: For, if it have any, this length muſt either be part of the length of the line AB, or of the line KB. It is not part of the length of KB; for, if the line KB be removed from AB, the point B which is the extremity of the line AB remains the ſame as it was: Nor is it part of the length of the line AB; for, if AB be removed from the line KB, the point B, which is the extremity of the line KB, does nevertheleſs remain: Therefore the point B has no length: And becauſe a point is in a line, and a line has neither breadth nor thickneſs, therefore a point has no length, breadth, nor thickneſs. And in this manner the definitions of a point, line, and ſuperficies are to be underſtood.



DEF. VII. B. I.

Inſtead of this definition as it is in the Greek copies, a more diſtinct one is given from a property of a plane ſuperficies, which is manifeſtly ſuppoſed in the elements, viz. that a ſtraight line drawn from any point in a plane to any other in it, is wholly in that plane.

DEF. VIII. B. I.

It ſeems that he who made this definition deſigned that it ſhould comprehend not only a plane angle contained by two ſtraight lines, but likewiſe the angle which ſome conceive to be made by a ſtraight line and a curve, or by two curve lines which meet one another in a plane: But, tho' the meaning of

he words ἐπ' εὐθείας, that is, in a ſtraight line, or in the ſame direction, be plain, when two ſtraight lines are ſaid to be in a ſtraight line, it does not appear what ought to be underſtood by theſe words, when a ſtraight line and a curve, or two curve lines, are ſaid to be in the ſame direction: at leaſt it cannot be explained in this place; which makes it probable that this definition, and that of the angle of a ſegment, and what is ſaid of the angle of a ſemicircle, and the angles of ſegments, in the 16. and 31. propoſitions of book 3. are the additions of ſome leſs ſkilful editor: On which account, eſpecially ſince they are quite uſeleſs, theſe definitions are diſtinguiſhed from the reſt by inverted double commas.

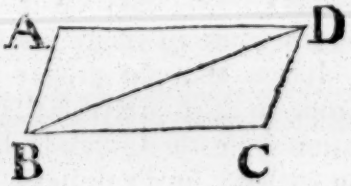
DEF. XVII. B. I.

The words, "which alſo divides the circle into two equal parts," are added at the end of this definition in all the copies, but are now left out as not belonging to the definition, being only a corollary from it. Proclus demonſtrates it by conceiving one of the parts into which the diameter divides the circle, to be applied to the other; for it is plain they muſt coincide, elſe the ſtraight lines from the center to the circumference would not be all equal: The ſame thing is eaſily deduced from the 1. prop. of book 3. and the 24. of the ſame; from the firſt of which it follows that ſemicircles are ſimilar ſegments of a circle. And from the other, that they are equal to one another.

DEF XXXIII. B. I.

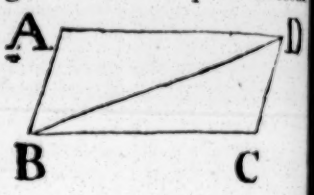
This definition has one condition more than is neceſſary; becauſe every quadrilateral figure which has its oppoſite ſides equal to one another, has likewiſe its oppoſite angles equal; and on the contrary.

Let ABCD be a quadrilateral figure, of which the oppoſite ſides AB, CD are equal to one another; as alſo AD and BC: Join BD; the two ſides AD, DB are equal to the two CB, BD, and the baſe AB is equal to the baſe CD; therefore by prop. 8. of book 1. the angle ADB is equal to the angle CBD; and by prop. 4. B. 1. the angle BAD is equal to the angle DCB, and ABD to BDC; and therefore alſo the angle ADC is equal to the angle ABC.



Book I.

And if the angle BAD be equal to the oppoſite angle BCD, and the angle ABC to ADC; the oppoſite ſides are equal; Becauſe, by prop 32. B. 1. all the angles of the quadrilateral figure ABCD are together equal to four right angles, and the two angles BAD, ADC are together equal to the two angles BCD, ABC: Wherefore BAD, ADC are the half of all the four angles; that is, BAD and ADC are equal to two right angles: And therefore AB, CD are parallels by prop. 28. B. 1. In the ſame manner AD, BC are parallels: Therefore ABCD is a parallelogram, and its oppoſite ſides are equal by 34. prop. B. 1.



PROP. VII. B. I.

There are two caſes of this propoſition, one of which is not in the Greek text, but is as neceſſary as the other: And that the caſe left out has been formerly in the text appears plainly from this, that the ſecond part of prop. 5 which is neceſſary to the demonſtration of this caſe, can be of no uſe at all in the elements, or any where elſe, but in this demonſtration; becauſe the ſecond part of prop. 5. clearly follows from the firſt part and prop. 13. B. 1. This part muſt therefore have been added to prop. 5 upon account of ſome propoſition betwixt the 5 and 13. but none of theſe ſtand in need of it except the 7. propoſition, on account of which it has been added: Beſides, the tranſlation from the Arabic has this caſe explicitly demonſtrated. And Proclus acknowledges that the ſecond part of prop. 5 was added upon account of prop. 7. but gives a ridiculous reaſon for it, "that it might afford an anſwer to objections made "againſt the 7." as if the caſe of the 7. which is left out, were as he expreſsly makes it, an objection againſt the propoſition itſelf. Whoever is curious may read what Proclus ſays of this in his commentary on the 5. and 7. propoſitions; for it is not worth while to relate his trifles at full length.

It was thought proper to change the enuntiation of this prop. ſo as to preſerve the very ſame meaning; the literal tranſlation from the Greek being extremely harſh, and difficult to be underſtood by beginners.

PRO

PROP. XI. B. I.

A corollary is added to this propoſition, which is neceſſary to prop. 1. B. 11. and otherwiſe.

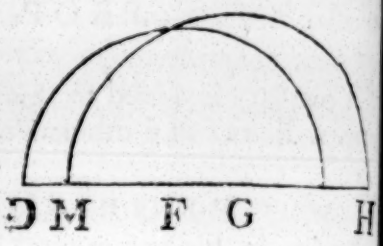
PROP. XX. and XXI. B. I.

Proclus, in his commentary, relates that the Epicureans de-ided this propoſition, as being manifeſt even to aſſes, and need-ing no demonſtration; and his anſwer is, that, though the truth of it be manifeſt to our ſenſes, yet it is ſcience which muſt give the reaſon why two ſides of a triangle are greater than the third: But the right anſwer to this objection againſt this and the 21. and ſome other plain propoſitions, is, that the number of axioms ought not to be encreaſed without neceſſity, as it muſt be if theſe propoſitions be not demonſtrated. Monſ. Clairault, in the pre-face to his elements of geometry, publiſhed in French at Paris, ann. 1741, ſays, That Euclid has been at the pains to prove, that the two ſides of a triangle which is included within another are together leſs than the two ſides of the triangle which in-cludes it; but he has forgot to add this condition, viz. that the triangles muſt be upon the ſame baſe; becauſe, unleſs this be added, the ſides of the included triangle may be greater than the ſides of the triangle which includes it, in any ratio which is leſs than that of two to one, as Pappus Alexandrinus has de-monſtrated in prop. 3. B. 3. of his mathematical collections.

PROP. XXII. B. I.

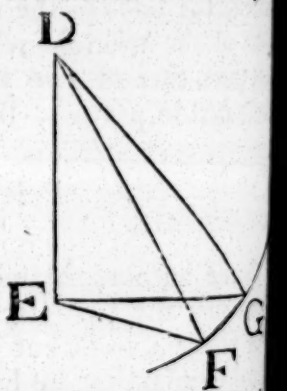
Some authors blame Euclid becauſe he does not demon-ſtrate that the two circles made uſe of in the conſtruction of this problem muſt cut one another: But this is very plain from the determination he has given, viz. that any two of the ſtraight lines DF, FG, GH muſt be great-er than the third: For, who is ſo dull, tho' only beginning to learn the elements, as not to perceive that the circle deſcribed from the center F, at the diſtance FD, muſt meet FH betwixt F and H, becauſe FD is leſs than FH; and that, for the like reaſon, the circle deſcribed from the center G, at the diſtance GH or GM, muſt meet DG betwixt D

Book I. and G; and that theſe circles muſt meet one another, becauſe FD and GH are together greater than FG? And this determination is eaſier to be underſtood than that which Mr Thomas Simpſon derives from it, and puts inſtead of Euclid's, in the 49. page of his elements of geometry, that he may ſupply the omiſſion he blames Euclid for; which determination is, that any of the three ſtraight lines muſt be leſs than the ſum, but greater than the difference of the other two: From this he ſhews the circles muſt meet one another, in one caſe; and ſays that it may be proved after the ſame manner in any other caſe: But the ſtraight line GM which he bids take from GF may be greater than it, as in the figure here annexed, in which caſe his demonſtration muſt be changed into another.



PROP. XXIV. B. I.

To this is added "of the two ſides DE, DF, let DE be "that which is not greater than the other;" that is, take that ſide of the two DE, DF which is not greater than the other, in order to make with it the angle EDG equal to BAC: Becauſe, without this reſtriction, there might be three different caſes of the propoſition, as Campanus and others make.



Mr Thomas Simpſon, in p. 262. of the ſecond edition of his elements of geometry, printed ann. 1760, obſerves, in his notes, that it ought to have been ſhewn that the point F falls below the line EG; this probably Euclid omitted, as it is very eaſy to perceive that DG being equal to DF, the point G is in the circumference of a circle deſcribed from the center D at the diſtance DF, and muſt be in that part of it which is above the ſtraight line EF, becauſe DG falls above DF, the angle EDG being greater than the angle EDF.

PROP. XXIX. B. I.

The propoſition which is uſually called the 5. poſtulate, 11. axiom, by ſome the 12. on which this 29. depends, has

given a great deal to do, both to ancient and modern geometers: It ſeems not to be properly placed among the Axioms, as, indeed, it is not ſelf-evident; but it may be demonſtrated thus.

DEFINITION 1.

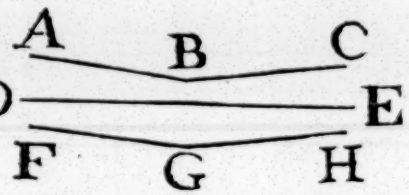
The diſtance of a point from a ſtraight line, is the perpendicular drawn to it from the point.

DEF. 2.

One ſtraight line is ſaid to go nearer to, or further from another ſtraight line, when the diſtance of the points of the firſt from the other ſtraight line become leſs or greater than they were; and two ſtraight lines are ſaid to keep the ſame diſtance from one another, when the diſtance of the points of one of them from the other is always the ſame.

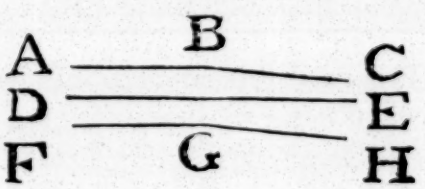
AXIOM.

A ſtraight line cannot firſt come nearer to another ſtraight line, and then go further from it, before it cuts it; and, in like manner, a ſtraight line cannot go further from another ſtraight line, and then come nearer to it; nor can a ſtraight line keep the ſame diſtance from another ſtraight line, and then come nearer to it, or go further from it; for a ſtraight line keeps always the ſame direction.



For example, the ſtraight line ABC cannot firſt come nearer to the ſtraight line DE, as from the point A to the point B, and then, from the point B to the point C, go further from the ſame DE: And, in like manner, the ſtraight line FGH cannot go further from DE, as from F to G, and then, from G to H, come nearer to the ſame DE: And ſo in the laſt caſe, as in fig. 2.

See the figure above.

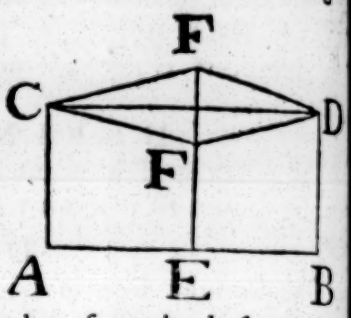


PROP. 1.

If two equal ſtraight lines AC, BD be each at right angles to the ſame ſtraight line AB; if the points C, D be joined by the ſtraight line CD, the ſtraight line EF drawn from any point E in AB unto CD, at right angles to AB, ſhall be equal to AC, or BD.

If EF be not equal to AC, one of them muſt be greater than the other; let AC be the greater; then, becauſe FE is

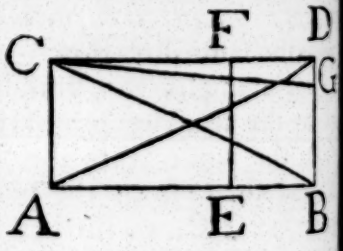
Book I. leſs than CA, the ſtraight line CFD is nearer to the ſtraight line AB at the point F than at the point C, that is, CF comes nearer to AB from the point C to F: But becauſe DB is greater than FE, the ſtraight line CFD is further from AB at the point D than at F, that is, FD goes further from AB from F to D: Therefore the ſtraight line CFD firſt comes nearer to the ſtraight line AB, and then goes further from it, before it cuts it, which is impoſſible. If FE be ſaid to be greater than CA, or DB, the ſtraight line CFD firſt goes further from the ſtraight line AB, and then comes nearer to it, which is alſo impoſſible. Therefore FE is not unequal to AC, that is, it is equal to it.



PROP. 2.

If two equal ſtraight lines AC, BD be each at right angles to the ſame ſtraight line AB; the ſtraight line CD which joins their extremities makes right angles with AC and BD.

Join AD, BC; and becauſe, in the triangles CAB, DBA, CA, AB are equal to DB, BA, and the angle CAB equal to
[a] 4. 1. the angle DBA; the baſe BC is equal [a] to the baſe AD: And in the triangles ACD, BDC, AC, CD are equal to BD, DC, and the baſe AD is equal to the baſe BC; therefore the angle ACD is e-
[b] 8. 1. qual [b] to the angle BDC: From any point E in AB draw EF unto CD, at right angles to AB; therefore, by Prop. 1. EF is equal to AC, or BD; wherefore, as has been juſt now ſhewn, the angle ACF is equal to the angle EFC: In the ſame manner, the angle BDF is equal to the angle EFD: but the angles ACD, BDC are equal; therefore the angles EFC and EFD are equal, and right
[c] 10. def. 1. angles [c]; wherefore alſo the angles ACD, BDC are right angles.



Cor. Hence, if two ſtraight lines AB, CD be at right angles to the ſame ſtraight line AC, and if betwixt them a ſtraight line BD be drawn at right angles to either of them, as to AB; then BD is equal to AC, and BDC is a right angle.

If AC be not equal to BD, take BG equal to AC, and join CG: Therefore, by this Propoſition, the angle ACG is a right angle; but ACD is alſo a right angle; wherefore the an-

gles

les ACD, ACG are equal to one another, which is impoſſible. Therefore BD is equal to AC; and by this Propoſition BDC is right angle. Book I.

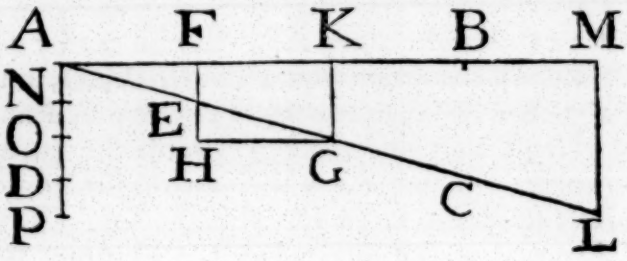
PROP. 3.

If two ſtraight lines which contain an angle be produced, here may be found in either of them a point from which the erpendicular drawn to the other ſhall be greater than any given raight line.

Let AB, AC be two ſtraight lines which make an angle with ne another, and let AD be the given ſtraight line; a point ay be found either in AB or AC, as in AC, from which the erpendicular drawn to the other AB ſhall be greater than AD.

In AC take any point E, and draw EF perpendicular to B; produce AE to G, ſo that EG be equal to AE; and oduce FE to H, and make EH equal to FE, and join HG. ecauſe, in the triangles AEF, GEH, AE, EF are equal to E, EH, each to each, and contain equal [a] angles, the angle HE is therefore equal [b] to the angle AFE which is a right an- e: Draw GK perpendicular to AB; and becauſe the ſtraight nes FK, HG e at right an- es to FH, and G at right an- es to FK; KG equal to FH, Cor. Pr. 2. at is, to the uble of FE.

[a] 15. 1.
[b] 4. 1.



the ſame manner, if AG be produced to L, ſo that GL be ual to AG, and LM be drawn perpendicular to AB, then M is double of GK, and ſo on. In AD take AN equal to , and AO equal to KG, that is, to the double of FE, or N; alſo take AP equal to LM, that is, to the double of KG, AO; and let this be done till the ſtraight line taken be great- than AD: Let this ſtraight line ſo taken be AP, and becauſe P is equal to LM, therefore LM is greater than AD. Which s to be done.

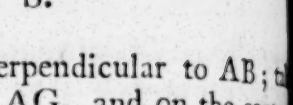
PROP. 4.

If two ſtraight lines AB, CD make equal angles EAB, CD with another ſtraight line EAC towards the ſame parts of AB and CD are at right angles to ſome ſtraight line.

Biſect

Book I.

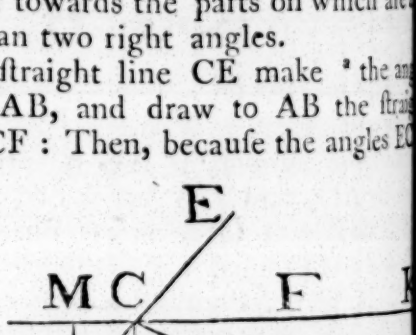
Biſect AC in F, and draw FG perpendicular to AB; th
CH in the ſtraight line CD equal to AG, and on the contra
ſide of AC to that on which AG is, and join FH: Therefo
in the triangles AFG, CFH the ſides FA, AG are equal
FC, CH, each to each, and the angle
[a] 15. 1. FAG, that [a] is, EAB is equal to the
[b] 4. 1. angle FCH; wherefore [b] the angle
AGF is equal to CHF, and AFG to
the angle CFH: To theſe laſt add the
common angle AFH; therefore the
two angles AFG, AFH are equal to
the two angles CFH, HFA, which
two laſt are equal together to two
[c] 13. 1. right angles [c], therefore alſo AFG,
[d] 14. 1. AFH are equal to two right angles, and conſequently [d] G
and FH are in one ſtraight line. And becauſe AGF is a ri
angle, CHF which is equal to it is alſo a right angle: Therefo
the ſtraight lines AB, CD are at right angles to GH.



PROP. 5.

If two ſtraight lines AB, CD be cut by a third ACE ſo
to make the interior angles BAC, ACD, on the ſame ſide
it, together leſs than two right angles; AB and CD being p
duced ſhall meet one another towards the parts on which are
two angles which are leſs than two right angles.

[a] 23. 1. At the point C in the ſtraight line CE make [a] the an
ECF equal to the angle EAB, and draw to AB the ſtrai
line CG at right angles to CF: Then, becauſe the angles EC
EAB are equal to one an-
other, and that the angles
ECF, FCA are together e-
[b] 13. 1. qual [b] to two right angles,
the angles EAB, FCA are
equal to two right angles.
But, by the hypotheſis, the
angles EAB, ACD are to-
gether leſs than two right
angles; therefore the angle
FCA is greater than ACD,
and CD falls between CF and AB: And becauſe CF and
make an angle with one another, by Prop. 3. a point may
found in either of them CD from which the perpendicu
drawn to CF ſhall be greater than the ſtraight line CG:



is point be H, and draw HK perpendicular to CF meeting AB in L: And becauſe AB, CF contain equal angles with AC on the ſame ſide of it, by Prop. 4. AB and CF are at right angles to the ſtraight line MNO which biſects AC in N and is perpendicular to CF: Therefore, by Cor. Prop. 2 CG and KL which are at right angles to CF are equal to one another: And HK is greater than CG, and therefore is greater than KL, and conſequently the point H is in KL produced. Wherefore the ſtraight line CDH drawn betwixt the points C, H which are on contrary ſides of AL, muſt neceſſarily cut the ſtraight line AB. Book I.

PROP. XXXV. B. I.

The demonſtration of this Propoſition is changed, becauſe, if the method which is uſed in it was followed, there would be three caſes to be ſeparately demonſtrated, as is done in the tranſlation from the Arabic; for, in the Elements, no caſe of a Propoſition that requires a different demonſtration, ought to be omitted. On this account, we have choſen the method which Monſ. Clairault has given, the firſt of any, as far as I know, in his Elements, page 21. and which afterwards Mr Simpſon gives in his, page 32. But whereas Mr Simpſon makes uſe of Prop. 26. B. 1. from which the equality of the two triangles does not immediately follow, becauſe, to prove that, the 4. of B. 1. muſt likewiſe be made uſe of, as may be ſeen, in the very ſame caſe, in the 34. Prop. B. 1. it was thought better to make uſe only of the 4. of B. 1.

PROP. XLV. B. I.

The ſtraight line KM is proved to be parallel to FL from the 33. Prop.; whereas KH is parallel to FG by conſtruction, and KHM, FGL have been demonſtrated to be ſtraight lines. A corollary is added from Commandine, as being often uſed.

PROP. XIII. B. II.

IN this Propoſition only acute angled triangles are mentioned, whereas it holds true of every triangle: And the demonſtrations of the caſes omitted are added; Commandine and Flavius have likewiſe given their demonſtrations of theſe caſes. Book II.

PROP. XIV. B. II.

In the demonſtration of this, ſome Greek editor has ignorantly inſerted the words, "but if not, one of the two BE,

"ED

Book II. " ED is the greater; let BE be the greater, and produce it t
" F," as if it was of any conſequence whether the greater
leſſer be produced: Therefore, inſtead of theſe words, the
ought to be read only, " but if not, produce BE to F."

PROP. I. B. III.

Book III. SEVERAL authors, eſpecially among the modern mathe
maticians and logicians, inveigh too ſeverely againſt indi
rect or Apagogic demonſtrations, and ſometimes ignorantl
enough; not being aware that there are ſome things that ca
not be demonſtrated any other way: Of this the preſent pro
poſition is a very clear inſtance, as no direct demonſtratio
can be given of it: Becauſe, beſides the definition of a circl
there is no principle or property relating to a circle anteceden
to this problem, from which either a direct or indirect de
monſtration can be deduced: Wherefore it is neceſſary that th
point found by the conſtruction of the problem be proved
be the center of the circle, by the help of this definition, an
ſome of the preceeding propoſitions: And becauſe, in the de
monſtration, this propoſition muſt be brought in, viz. ſtraigh
lines from the center of a circle to the circumference are equa
and that the point found by the conſtruction cannot be aſſu
med as the center, for this is the thing to be demonſtrated;
is manifeſt ſome other point muſt be aſſumed as the center
and if from this aſſumption an abſurdity follows, as Euclid de
monſtrates there muſt, then it is not true that the point aſſume
is the center; and as any point whatever was aſſumed, it follow
that no point, except that found by the conſtruction, can be th
center, from which the neceſſity of an indirect demonſtration i
this caſe is evident.

PROP. XIII. B. III.

As it is much eaſier to imagine that two circles may touc
one another within in more points than one, upon the ſam
ſide, than upon oppoſite ſides; the figure of that caſe ough
not to have been omitted; but the conſtruction in the Gre
text would not have ſuited with this figure ſo well, becauſe th
centers of the circles muſt have been placed near to the ci
cumferences: On which account another conſtruction and de
monſtration is given, which is the ſame with the ſecond pa
of that which Campanus has tranſlated from the Arab
whi

here without any reaſon the demonſtration is divided into two parts. Book III.

PROP. XV. B. III.

The converſe of the ſecond part of this propoſition is wanting, tho' in the preceeding, the converſe is added, in a like caſe, both in the enunciation and demonſtration; and it is now added in this. Beſides, in the demonſtration of the firſt part of this 15th, the diameter AD (ſee Commandine's figure) is proved to be greater than the ſtraight line BC by means of another ſtraight line MN; whereas it may be better done without it: on which accounts, we have given a different demonſtration, like to that which Euclid gives in the preceeding 14th, and to that which Theodoſius gives in prop. 6. B. 1. of his Spherics, in this very affair.

PROP. XVI. B. III.

In this we have not followed the Greek nor the Latin tranſlation literally, but have given what is plainly the meaning of the propoſition, without mentioning the angle of the ſemicircle, or that which ſome call the cornicular angle which they conceive to be made by the circumference and the ſtraight line which is at right angles to the diameter, at its extremity; which angles have furniſhed matter of great debate between ſome of the modern geometers, and given occaſion of deducing ſtrange conſequences from them, which are quite avoided by the manner in which we have expreſſed the propoſition. And in like manner, we have given the true meaning of prop. 31. b. 3. without mentioning the angles of the greater or leſſer ſegments: Theſe paſſages, Vieta, with good reaſon, ſuſpects to be adulterated, in the 386. page of his Oper. Math.

PROP. XX. B. III.

The firſt words of the ſecond part of this demonſtration, "κικλασθω δη παλιν," are wrong tranſlated by Mr Briggs and Dr Gregory "Rurſus inclinetur;" for the tranſlation ought to be "Rurſus inflectatur," as Commandine has it: A ſtraight line is ſaid to be inflected either to a ſtraight, or curve line, when a ſtraight line is drawn to this line from a point, and from the point in which it meets it, a ſtraight line making an angle with the former is drawn to another point, as is evident from the 90. prop. of Euclid's Data: for thus the whole line betwixt the firſt and laſt points, is inflected or broken at the

Book III. the point of inflection, where the two ſtraight lines meet. An in the like ſenſe two ſtraight lines are ſaid to be inflected fro two points to a third point, when they make an angle at th point; as may be ſeen in the deſcription given by Pappus A lexandrinus of Apollonius's Books de Locis planis, in the pr face to his 7. Book: We have made the expreſſion fuller fro the 90. prop. of the data.

PROP. XXI. B. III.

There are two caſes of this propoſition, the ſecond of whic viz. when the angles are in a ſegment not greater than a ſem circle, is wanting in the Greek: And of this a more ſimp demonſtration is given than that which is in Commandine, being derived only from the firſt caſe, without the help of t angles.

PROP. XXIII. and XXIV. B. III.

In Propoſition 24. it is demonſtrated, that the ſegment AE muſt coincide with the ſegment CFD (ſee Commandine's f gure) and that it cannot fall otherwiſe, as CGD, ſo as to c the other circle in a third point G, from this, that, if it did, circle could cut another in more points than two: But th ought to have been proved to be impoſſible in the 23. Prop. well as that one of the ſegments cannot fall within the othe This part then is left out in the 24. and put in its proper plac the 23d Propoſition.

PROP. XXV. B. III.

This propoſition is divided into three caſes, of which tw have the ſame conſtruction and demonſtration; therefore it now divided only into two caſes.

PROP. XXXIII. B. III.

This alſo in the Greek is divided into three caſes, of whic two, viz. one, in which the given angle is acute, and the other which it is obtuſe, have exactly the ſame conſtruction and d monſtration; on which account, the demonſtration of the caſe is left out as quite ſuperfluous, and the addition of ſom unſkilful editor; beſides the demonſtration of the caſe when t angle given is a right angle, is done a round about way, and therefore changed to a more ſimple one, as was done by Clavi

PRO

PROP. XXXV. B. III.

As the 25 and 33. propoſitions are divided into more caſes, this 35. is divided into fewer caſes than are neceſſary. Nor n it be ſuppoſed that Euclid omitted them becauſe they are ſy; as he has given the caſe, which, by far, is the eaſieſt of em all, viz. that in which both the ſtraight lines paſs through e center: And in the following propoſition he ſeparately de-onſtrates the caſe in which the ſtraight line paſſes through the nter, and that in which it does not paſs thro' the center: So t it ſeems Theon, or ſome other, has thought them too long inſert: But caſes that require different demonſtrations, ſhould be left out in the elements, as was before taken notice of: eſe caſes are in the tranſlation from the Arabic, and are now into the text.

PROP. XXXVII. B. III.

At the end of this the words, "in the ſame manner it may e demonſtrated, if the center be in AC," are left out as the ition of ſome ignorant editor.

DEFINITIONS of BOOK IV.

WHEN a point is in a ſtraight, or any other line, this point is by the Greek geometers ſaid απτεσθαι, to be n, or in that line, and when a ſtraight line or circle meets rcle any way, the one is ſaid απτεσθαι to meet the other: But en a ſtraight line or circle meets a circle ſo as not to cut it, ſaid εφαπτεσθαι, to touch the circle; and theſe two terms never promiſcuouſly uſed by them: Therefore, in the 5. de-tion of B. 4. the compound εφαπτηται muſt be read, inſtead he ſimple απτηται: And in the 1. 2. 3. and 6 definitions Commandine's tranſlation, "tangit," muſt be read inſtead "contingit:" And in the 2. and 3. definitions of Book 3. ſame change muſt be made: But, in the Greek text of pro-tions 11. 12. 13. 18. 19. B. 3. the compound verb is to ut for the ſimple.

PROP. IV. B. IV.

this, as alſo in the 8. and 13. propoſitions of this book, demonſtrated indirectly, that the circle touches a ſtraight ; whereas in the 17. 33. and 37. propoſitions of book he ſame thing is directly demonſtrated: And this way we have

Book IV. have choſen to uſe in the propoſitions of this book, as it ſhorter.

PROP. V. B. IV.

The demonſtration of this has been ſpoiled by ſome unſkil hand: For he does not demonſtrate, as is neceſſary, that the t ſtraight lines which biſect the ſides of the triangle at right angl muſt meet one another; and, without any reaſon, he divi the propoſition into three caſes; whereas, one and the ſame co ſtruction and demonſtration ſerves for them all, as Campa has obſerved; which uſeleſs repetitions are now left out: T Greek text alſo in the corollary is manifeſtly vitiated, wh mention is made of a given angle, though there neither is, n can be any thing in the propoſition relating to a given angle.

PROP. XV. and XVI. B. IV.

In the corollary of the firſt of theſe, the words equilat and equiangular are wanting in the Greek: And in prop. inſtead of the circle ABCD, ought to be read the circumfere ABCD: Where mention is made of its containing fifteen eq parts.

Book V.

DEF. III. B. V.

MANY of the modern mathematicians reject this definiti The very learned Dr Barrow has explained it at larg the end of his third lecture of the year 1666, in which alſo anſwers the objections made againſt it as well as the ſub would allow: And at the end gives his opinion upon the wh as follows:

"I ſhall only add, that the author had, perhaps, no
" ther deſign in making this definition, than (that he m
" more fully explain and embelliſh his ſubject) to give a g
" ral and ſummary idea of ratio to beginners, by premi
" this metaphyſical definition, to the more accurate de
" tions of ratios that are the ſame to one another, or on
" which is greater, or leſs than the other: I call it a m
" phyſical, for it is not properly a mathematical defini
" ſince nothing in mathematics depends on it, or is dedu
" nor, as I judge, can be deduced from it: And the de
" tion of analogy, which follows, viz. Analogy is the
" [illegible]

" litude of ratios, is of the ſame kind, and can ſerve for no " purpoſe in mathematics, but only to give beginners ſome " general, tho' groſs and confuſed notion of analogy: But the " whole of the doctrine of ratios, and the whole of mathema- " tics depend upon the accurate mathematical definitions which " follow this: To theſe we ought principally to attend, as the " doctrine of ratios is more perfectly explained by them; this " third, and others like it, may be intirely ſpared without any " loſs to geometry: As we ſee in the 7th book of the elements, " where the proportion of numbers to one another is defined, " and treated of, yet without giving any definition of the ratio " of numbers; tho' ſuch a definition was as neceſſary and uſe- " ful to be given in that book, as in this: But indeed there is " ſcarce any need of it in either of them: Though I think that " a thing of ſo general and abſtracted a nature, and thereby the " more difficult to be conceived, and explained, cannot be more " commodiouſly defined, than as the author has done: Upon " which account I thought fit to explain it at large, and defend " it againſt the captious objections of thoſe who attack it." To his citation from Dr Barrow I have nothing to add, except that I fully believe the 3. and 8. definitions are not Euclid's, but added by ſome unſkilful editor.

DEF. XI. B. V.

It was neceſſary to add the word " continual" before " pro- " portionals" in this definition; and thus it is cited in the 33. prop. of book 11.

After this definition ought to have followed the definition of compound ratio, as this was the proper place for it; duplicate and triplicate ratio being ſpecies of compound ratio. But Theon has made it the 5. def. of B. 6. where he gives an abſurd and intirely uſeleſs definition of compound ratio: For this reaſon we have placed another definition of it betwixt the 11. and 12. of this book, which, no doubt, Euclid gave; for he cites it expreſs- ly in prop. 23. B. 6. and which Clavius, Herigon, and Barrow have likewiſe given, but they retain alſo Theon's, which they ought to have left out of the elements.

DEF. XIII. B. V.

This and the reſt of the definitions following, contain the ex- plication of ſome terms which are uſed in the 5. and following books; which, except a few, are eaſily enough underſtood from

Book V. the propoſitions of this book where they are firſt mentioned. They ſeem to have been added by Theon, or ſome other However it be, they are explained ſomething more diſtinctly for the ſake of learners.

PROP. IV. B. V.

In the conſtruction preceeding the demonſtration of this the words ἀ ἐτυχε, any whatever, are twice wanting in the Greek, as alſo in the Latin tranſlations; and are now added, being wholly neceſſary.

Ibid. in the demonſtration; in the Greek, and in the Latin tranſlation of Commandine, and in that of Mr Henry Briggs which was publiſhed at London in 1620, together with the Greek text of the firſt ſix books, which tranſlation in this place is followed by Dr Gregory in his edition of Euclid, there is the ſentence following, viz. "and of A and C have been taken equimultiples K, L; and of B and D, any equimultiples whatever (ἀ ἐτυχε) M, N;" which is not true, the words "any whatever," ought to be left out: And it is ſtrange that neither Mr Briggs, who did right to leave out theſe words in one place of prop. 13. of this book, nor Dr Gregory, who changed them into the word "ſome" in three places, and left them out in a fourth of that ſame prop. 13. did not alſo leave them out in this place of prop. 4 and in the ſecond of the two places where they occur in prop. 17. of this book, in neither of which they can ſtand conſiſtent with truth: And in none of all the places, even in thoſe which they corrected in their Latin tranſlation, have they cancelled the words ἀ ἐτυχε in the Greek text as they ought to have done.

The ſame words ἀ ἐτυχε are found in four places of prop. 11. of this book, in the firſt and laſt of which, they are neceſſary, but in the ſecond and third, though they are true, they are quite ſuperfluous; as they likewiſe are in the ſecond of the two places in which they are found in the 12. prop. and in the like places of prop. 22. 23. of this book: But are wanting in the laſt place of prop. 23. as alſo in prop. 25. Book 11.

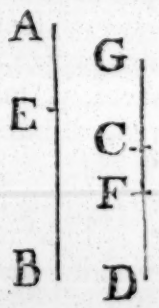
COR. IV. PROP. B. V.

This corollary has been unſkilfully annexed to this propoſition, and has been made inſtead of the legitimate demonſtration, which, without doubt, Theon, or ſome other editor has taken away, not from this, but from its proper place

his book: The author of it deſigned to demonſtrate, that if four magnitudes E, G, F, H be proportionals, they are alſo proportionals inverſely; that is, G is to E, as H to F; which is true, but the demonſtration of it does not in the leaſt depend upon this 4th prop. or its demonſtration: For, when he ſays, 'becauſe it is demonſtrated that if K be greater than M, L is 'greater than N,' &c. This indeed is ſhewn in the demonſtration of the 4th prop. but not from this that E, G, F, H are proportionals; for this laſt is the concluſion of the propoſition. Wherefore theſe words, "becauſe it is demonſtrated," &c. are wholly foreign to his deſign: And he ſhould have proved, that if K be greater than M, L is greater than N, from this, that E, G, F, H are proportionals, and from the 5th def. of this book, which he has not; but is done in propoſition B, which we have given, in its proper place, inſtead of this corollary; And another corollary is placed after the 4th prop. which is often of uſe; and is neceſſary to the demonſtration of prop. 18. of this book.

PROP. V. B. V.

In the conſtruction which precedes the demonſtration of this propoſition, it is required that EB may be the ſame multiple of CG, that AE is of CF; that is, that EB be divided into as many equal parts, as there are parts in AE equal to CF: From which it is evident, that this conſtruction is not Euclid's; for he does not ſhew the way of dividing ſtraight lines, and far leſs other magnitudes, into any number of equal parts, until the 9th propoſition of B. 6. and he never requires any thing to be done in the conſtruction, of which he had not before given the method of doing: For this reaſon, we have changed the conſtruction to one, which, without doubt, is Euclid's, in which nothing is required but to add a magnitude to itſelf a certain number of times; and this is to be found in the tranſlation from the Arabic, tho' the enunciation of the propoſition and the demonſtration are there very much ſpoiled. Jacobus Peletarius, who was the firſt, as far as I know, who took notice of this error, gives alſo the right conſtruction in his edition of Euclid, after he had given the other which he blames: he ſays, he would not leave it out, becauſe it was fine, and might ſharpen one's genius to invent others like it; whereas there



Book V. is not the leaſt difference between the two demonſtrations, except a ſingle word in the conſtruction, which very probably has been owing to an unſkilful Librarian. Clavius likewiſe gives both the ways; but neither he nor Peletarius takes notice of the reaſon why the one is preferable to the other.

PROP. VI. B. V.

There are two caſes of this propoſition, of which only the firſt and ſimpleſt is demonſtrated in the Greek: And it is probable Theon thought it was ſufficient to give this one, ſince he was to make uſe of neither of them in his mutilated edition of the 5th book; and he might as well have left out the other, as alſo the 5th propoſition for the ſame reaſon: The demonſtration of the other caſe is now added, becauſe both of them, as alſo the 5th propoſition, are neceſſary to the demonſtration of the 18th propoſition of this book. The tranſlation from the Arabic gives both caſes briefly.

PROP. A. B. V.

This propoſition is frequently uſed by geometers, and it is neceſſary in the 25th prop. of this book, 31. of the 6. and 34. of the 11 and 15. of the 12th book: It ſeems to have been taken out of the elements by Theon, becauſe it appeared evident enough to him, and others who ſubſtitute the confuſed and indiſtinct idea the vulgar have of proportionals, in place of that accurate idea which is to be got from the 5th def. of this book. Nor can there be any doubt that Eudoxus or Euclid gave it a place in the elements, when we ſee the 7th and 9th of the ſame book demonſtrated, tho' they are quite as eaſy and evident as this. Alphonſus Borellus takes occaſion from this propoſition to cenſure the 5th definition of this book very ſeverely, but moſt unjuſtly: In page 126. of his Euclid reſtored, printed at Piſa in 1658, he ſays, "Nor can even the "leaſt degree of knowledge be obtained from the foreſaid "property," viz. that which is contained in 5th def. 5. "That "if four magnitudes be proportionals, the third muſt neceſ- "ſarily be greater than the fourth, when the firſt is greater "than the ſecond; as Clavius acknowledges in the 16th prop. "of the 5th book of the elements." But though Clavius makes no ſuch acknowledgement expreſsly, he has given Borellus a handle to ſay this of him; becauſe, when Clavius, in the above cited place, cenſures Commandine, and that very juſtly, for demonſtrating this propoſition by help of the 16th of the 5th, yet he himſelf gives no demonſtration of it, but thinks it plain from

rom the nature of proportionals, as he writes in the end of the 4th and 16th prop. B. 5. of his edition, and is followed by Herigon in Schol. 1. prop. 14. B. 5. as if there was any nature of proportionals antecedent to that which is to be derived and underſtood from the definition of them: And indeed, though it is very eaſy to give a right demonſtration of it, no body, as far as I know, has given one, except the learned Dr Barrow, who, in anſwer to Borellus's objection, demonſtrates it indirectly, but very briefly and clearly, from the 5th definition, in the 322d page of his Lect. Mathem. from which definition it may alſo be eaſily demonſtrated directly: On which account we have placed it next to the propoſitions concerning equimultiples.

PROP. B. BOOK V.

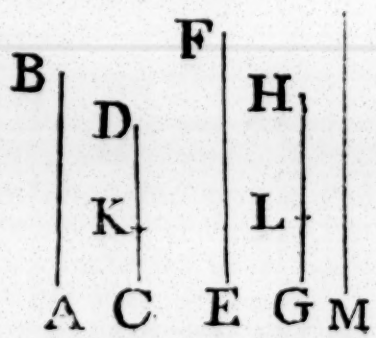
This alſo is eaſily deduced from the 5th def. B. 5. and therefore is placed next to the other, for it was very ignorantly made a corollary from the 4th prop. of this book. See the note on that corollary.

PROP. C. B. V.

This is frequently made uſe of by geometers, and is neceſſary to the 5th and 6th propoſitions of the 10th book. Clavius in his notes ſubjoined to the 8th def. of book 5. demonſtrates it only in numbers, by help of ſome of the propoſitions of the 7th book, in order to demonſtrate the property contained in the 5th definition of the 5th book, when applied to numbers, from the property of proportionals contained in the 20th def. of the 7th book: And moſt of the commentators judge it difficult to prove that four magnitudes which are proportionals according to the 20th def. of 7th B, are alſo proportionals according to the 5th def. of 5th book: But this is eaſily made out, as follows:

Firſt, If A, B, C, D be four magnitudes, ſuch that A is the ſame multiple, or the ſame part of B, which C is of D; A, B, C, D are proportionals: This is demonſtrated in propoſition C.

Secondly, If AB contain the ſame parts of CD that EF does of GH; in this caſe likewiſe AB is to CD, as EF to GH.



Book V.

Let CK be a part of CD, and GL the ſame part of GH and let AB be the ſame multiple of CK, that EF is of GL: Therefore, by prop. C. of 5th book, AB is to CK, as EF to GL: And CD, GH are equimultiples of CK, GL the ſecond and fourth; wherefore, by Cor. prop. 4. Book 5. AB is to CD, as EF to GH.

And if four magnitudes be proportionals according to the 5th def of Book 5. they are alſo proportionals according to the 20th def. of Book 7.

Firſt, If A be to B, as C to D; then if A be any multiple or part of B, C is the ſame multiple or part of D, by prop. D. of B. 5.

Next, If AB be to CD, as EF to GH; then if AB contains any parts of CD, EF contains the ſame parts of GH: For let CK be a part of CD, and GL the ſame part of GH, and let AB be a multiple of CK; EF is the ſame multiple of GL: Take M the ſame multiple of GL that AB is of CK; therefore by prop. C. of B. 5. AB is to CK, as M to GL; and CD, GH are equimultiples of CK, GL; wherefore by Cor. prop. 4. B. 5. AB is to CD, as M to GH: And, by the hypotheſis, AB is to CD, as EF to GH; therefore M is equal to EF by prop. 9. Book 5. and conſequently EF is the ſame multiple of GL that AB is of CK.

PROP. D. B. V.

This is not unfrequently uſed in the demonſtration of other propoſitions, and is neceſſary in that of prop. 9. B. 6. It ſeems Theon has left it out for the reaſon mentioned in the notes at prop. A.

PROP. VIII. B. V.

In the demonſtration of this, as it is now in the Greek, there are two caſes, (ſee the demonſtration in Hervagius, or Dr Gregory's edition), of which the firſt is that in which AE is leſs than EB; and in this, it neceſſarily follows that HΘ the multiple of EB is greater than ZH the ſame multiple of AE, which laſt multiple, by the conſtruction, is greater than Δ; whence alſo HΘ muſt be greater than Δ: But, in the ſecond caſe, viz. that in which EB is leſs than AE, tho' ZH be greater than Δ, yet HΘ may be leſs than the ſame Δ; ſo that there cannot be taken a multiple of Δ which is the firſt that is greater

greater than K, or HΘ, becauſe Δ itſelf is greater than it: Upon this account, the author of this demonſtration found it neceſſary to change one part of the conſtruction that was made uſe of in the firſt caſe: But he has, without any neceſſity, changed alſo another part of it, viz. when he orders to take N that multiple of Δ which is the firſt that is greater than ZH; for he might have taken that multiple of Δ which is the firſt that is greater than HΘ, or K, as was done in the firſt caſe: He likewiſe brings in this K into the demonſtration of both caſes, without any reaſon, for it ſerves to no purpoſe but to lengthen the demonſtration. There is alſo a

third caſe, which is not mentioned in this demonſtration, viz. that in which AE in the firſt, or EB in the ſecond of the two other caſes, is greater than D; and in this any equimultiples, as the doubles, of AE, KB are to be taken, as is done in this edition, where all the caſes are at once demonſtrated: And from this it is plain that Theon, or ſome other unſkilful editor has viciated this propoſition.

PROP. IX. B. V.

Of this there is given a more explicit demonſtration than that which is now in the elements.

PROP. X. B. V.

It was neceſſary to give another demonſtration of this propoſition, becauſe that which is in the Greek and Latin, or other editions, is not legitimate: For the words *greater*, *the ſame* or *equal*, *leſſer* have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5th and 7th definitions of Book 5. By the help of theſe let us examine the demonſtration of the 10th prop. which proceeds thus: "Let A "have to C a greater ratio, than B to C: I ſay that A is greater "than B. For if it is not greater, it is either equal, or leſs. "But A cannot be equal to B, becauſe then each of them "would have the ſame ratio to C; but they have not. Therefore A is not equal to B." The force of which reaſoning is this, if A had to C, the ſame ratio that B has to C, then if any equimultiples whatever of A and B be taken, and any

Book V.

multiple whatever of C; if the multiple of A be greater than the multiple of C, then, by the 5th def. of Book 5. the multiple of B is alſo greater than that of C: But, from the hypotheſis that A has a greater ratio to C, than B has to C, there muſt by the 7th def. of Book 5. be certain equimultiples of A and B and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than the ſame multiple of C: And this propoſition directly contradicts the preceeding; wherefore A is not equal to B. The demonſtration of the 10th prop. goes on thus: "But neither is A leſs than B; becauſe then A would have a leſs ratio to C, than B has to it: But it has not a leſs ratio, therefore A is not leſs than B," &c. Here it is ſaid that "A would have a leſs ratio to C, than B has to C," or, which is the ſame thing, that B would have a greater ratio to C than A to C; that is, by 7th def. Book 5. there muſt be ſome equimultiples of B and A, and ſome multiple of C ſuch, that the multiple of B is greater than the multiple of C, but the multiple of A is not greater than it: And it ought to have been proved that this can never happen if the ratio of A to C be greater than the ratio of B to C; that is, it ſhould have been proved, that, in this caſe, the multiple of A is always greater than the multiple of C, whenever the multiple of B is greater than the multiple of C: for, when this is demonſtrated, it will be evident that B cannot have a greater ratio to C, than A has to C, or, which is the ſame thing, that A cannot have a leſs ratio to C, than B has to C: But this is not at all proved in the 10th propoſition; but if the 10th were once demonſtrated, it would immediately follow from it, but cannot without it be eaſily demonſtrated, as he that tries to do it will find. Wherefore the 10th propoſition is not ſufficiently demonſtrated. And it ſeems that he who has given the demonſtration of the 10th propoſition as we now have it, inſtead of that which Eudoxus or Euclid had given, has been deceived in applying what is manifeſt, when underſtood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and leſs than another. That thoſe things which are equal to the ſame are equal to one another, is a moſt evident axiom when underſtood of magnitudes; yet Euclid does not make uſe of it to infer that thoſe ratios which are the ſame to the ſame ratio, are the ſame to one another; but explicitly demonſtrates this in prop. 11. of Book 5. The demonſtration we have given of the 10th prop. is

doubt the ſame with that of Eudoxus or Euclid, as it is immediately and directly derived from the definition of a greater o, viz the 7. of the 5.

The above mentioned propoſition, viz. If A have to C a ater ratio than B to C, and if of A and here be taken certain equimultiples, and e multiple of C; then if the multiple B be greater than the multiple of C, the tiple of A is alſo greater than the ſame, hus demonſtrated.

Let D, E be equimultiples of A, B, and multiple of C, ſuch, that E the multiple B is greater than F; D the multiple of is alſo greater than F.

Becauſe A has a greater ratio to C, than to C, A is greater than B, by the 10. p. B. 5. therefore D the multiple of is greater than E the ſame multiple of And E is greater than F; much more refore D is greater than F.

PROP. XIII. B. V.

n Commandine's, Briggs's, and Gregory's tranſlations, at the inning of this demonſtration, it is ſaid, "And the multiple of C is greater than the multiple of D; but the multiple of E is not greater than the multiple of F;" which ds are a literal tranſlation from the Greek: But the ſenſe ently requires that it be read, "ſo that the multiple of C greater than the multiple of D; but the multiple of E be ot greater than the multiple of F." And thus this place was ored to the true reading in the firſt editions of Commandine's Euclid printed in 8vo at Oxford; but, in the later editions, at leaſt in that of 1747, the error of the Greek text was t in.

There is a corollary added to Prop. 13. as it is neceſſary to 20. and 21. Prop. of this book, and is as uſeful as the poſition.

PROP. XIV. B. V.

The two caſes of this, which are not in the Greek, are added the demonſtration of them not being exactly the ſame with of the firſt caſe.

PROP.

PROP. XVII. B. V.

The order of the words in a clauſe of this is changed to a more natural: As was alſo done in prop. 11.

PROP. XVIII. B. V.

The demonſtration of this is none of Euclid's, nor is it legitimate; for it depends upon this hypotheſis, that to any two magnitudes, two of which, at leaſt, are of the ſame kind there may be a fourth proportional; which if not proved, the demonſtration now in the text is of no force: But this is aſſumed without any proof; nor can it, as far as I am able to diſcern, be demonſtrated by the propoſitions preceeding it: ſo far is it from deſerving to be reckoned an axiom, as Clavius, after other commentators, would have it, at the end of the definitions of the 5th book. Euclid does not demonſtrate it, nor does he ſhew how to find the fourth proportional, before the 12th prop. of the 6th Book: And he never aſſumes any thing in the demonſtration of a propoſition, which he had not before demonſtrated; at leaſt, he aſſumes nothing the exiſtence of which is not evidently poſſible; for a certain concluſion can never be deduced by the means of an uncertain propoſition. Upon this account, we have given a legitimate demonſtration of this propoſition inſtead of that in the Greek and other editions, which very probably Theon, at leaſt ſome other, put in the place of Euclid's, becauſe he thought it too proli. And as the 17th prop. of which this 18th is the converſe, is demonſtrated by help of the 1ſt and 2d propoſitions of this book ſo, in the demonſtration now given of the 18th, the 5th prop. and both caſes of the 6th are neceſſary, and theſe two propoſitions are the converſes of the 1ſt and 2d. Now the 5th and 6th do not enter into the demonſtration of any propoſition in this book as we now have it: Nor can they be of uſe in any propoſition of the Elements, except in this 18th, and this is a manifeſt proof, that Euclid made uſe of them in his demonſtration of it, and that the demonſtration now given, which is exactly the converſe of that of the 17th, as it ought to be, differs nothing from that of Eudoxus or Euclid: For the 5th, and 6th have undoubtedly been put into the 5th Book for the ſake of ſome propoſitions in it, as all the other propoſitions about equimultiples have been.

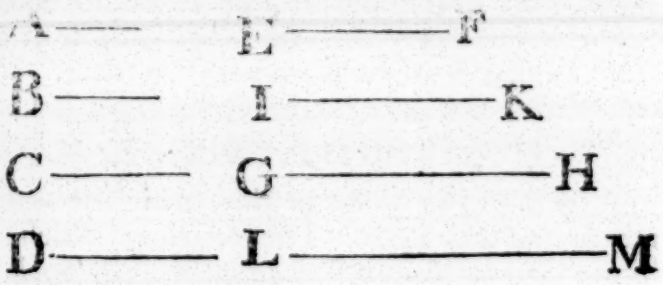
Hieronymus Saccherius, in his book named Euclides abſque omni nævo vindicatus, printed at Milan Ann. 1733, in 4to, knowledges

wledges, this blemiſh in the demonſtration of the 18. and he may remove it, and render the demonſtration we now e of it legitimate, he endeavours to demonſtrate the following poſition, which is in page 115. of his book, viz.

Let A, B, C, D be four magnitudes, of which the two irſt are of one kind, and alſo the two others either of the me kind with the two firſt, or of ſome other the ſame ind with one another. I ſay the ratio of the third C to the urth D, is either equal to, or greater, or leſs than the ratio f the firſt A to the ſecond B."

And after two propoſitions premiſed as Lemmas, he proceeds .

Either among all the poſſible equimultiples of the firſt A, and of the third C, and, at the ſame time, among all e poſſible equimultiples of the ſecond B, and of the urth D, there can be found ſome one multiple EF of the irſt A, and one IK of the ſecond B, that are equal to one nother; and alſo (in the ſame caſe) ſome one multiple GH of the third C equal to LM the multiple of the fourth D, or ſuch equality is no where to be found. If the firſt aſe happen,



i. e. if ſuch quality is to found] it is aniſeſt from hat is be- ore demon- rated, that A is to B, as C to D: But if ſuch ſimultaneous equality be not to be ound upon both ſides, it will be found either upon one ide, as upon the ſide of A [and B;] or it will be found pon neither ſide; if the firſt happen; therefore (from uclid's definition of greater and leſſer ratio foregoing) A has to B, a greater or leſs ratio than C to D; accordng as GH the multiple of the third C is leſs, or greater han LM the multiple of the fourth D: But if the ſecond aſe happen; therefore upon the one ſide, as upon the ſide f A the firſt and B the ſecond, it may happen that the ultiple EF, [viz. of the firſt] may be leſs than IK the ultiple of the ſecond, while, on the contrary, upon the oher ſide, [viz. of C and D] the multiple GH [of the third] is greater than the other multiple LM [of the fourth D:] And then (from the ſame definition of Euclid) the ra-

" tio

Book V. " tio of the firſt A to the ſecond B, is leſs than the ratio of
" third C to the fourth D; or on the contrary.

" Therefore the axiom, [i. e. the propoſition before ſet dow
" remains demonſtrated," &c.

Not in the leaſt; but it remains ſtill undemonſtrated: what he ſays may happen, may, in innumerable caſes, n happen; and therefore his demonſtration does not hold: example, if A be the ſide, and B the diameter of a ſqua and C the ſide, and D the diameter of another ſquare; t can in no caſe be any multiple of A equal to any of B; any one of C equal to one of D, as is well known; yet it can never happen that when any multiple of A is gre than a multiple of B, the multiple of C can be leſs than the tiple of D, nor when the multiple of A is leſs than that o the multiple of C can be greater than that of D, viz. tak equimultiples of A and C, and equimultiples of B and For A, B, C, D are proportionals; and ſo if the multipl A be greater, &c. than that of B, ſo muſt that of C be gre &c. than that of D; By 5th Def. B. 5.

The ſame objection holds good againſt the demonſtra which ſome give of the 1ſt prop. of the 6th book, which we h made againſt this of the 18th prop. becauſe it depends upon ſame inſufficient foundation with the other.

PROP. XIX. B. V.

A corollary is added to this, which is as frequently uſe the propoſition itſelf. The corollary which is ſubjoined in the Greek, plainly ſhews that the 5th book has been viti by editors who were not geometers: For the converſion ratios does not depend upon this 19th, and the demonſtra which ſeveral of the commentators on Euclid give of con ſion, is not legitimate, as Clavius has rightly obſerved, has given a good demonſtration of it which we have put in poſition E; but he makes it a corollary from the 19th, and gins it with the words, " Hence it eaſily follows," thoug does not at all follow from it.

PROP. XX. XXI. XXII. XXIII. XXIV. B. V.

The demonſtrations of the 20th and 21ſt propoſitions ſhorter than thoſe Euclid gives of eaſier propoſitions, e in the preceeding, or following books: Wherefore it was per to make them more explicit, and the 22d and 23d p ſitions are, as they ought to be, extended to any numb magnitu

nitudes: And, in like manner, may the 24. be, as is taken
ce of in a corollary; and another corollary is added, as
ul as the propoſition, and the words "any whatever" are ſup-
d near the end of prop. 23. which are wanting in the Greek
, and the tranſlations from it.

n a paper writ by Philippus Naudaeus, and publiſhed, after
death, in the hiſtory of the Royal Academy of Sciences of
in, ann. 1745, page 50. the 23d prop. of the 5th book is
ured as being obſcurely enunciated, and, becauſe of this,
ixly demonſtrated: The enunciation there given is not Eu-
's but Tacquet's, as he acknowledges, which, though not ſo
expreſſed, is, upon the matter, the ſame with that which
ow in the Elements. Nor is there any thing obſcure in
hough the author of the paper has ſet down the propor-
als in a diſadvantageous order, by which it appears to be
ure: But, no doubt, Euclid enunciated this 23d, as well as
22d, ſo as to extend it to any number of magnitudes, which,
n two and two, are proportionals, and not of ſix only; and
is general caſe the enunciation which Naudaeus gives, can-
be well applied.

he demonſtration which is given of this 23d, in that paper,
quite wrong; becauſe, if the proportional magnitudes be
e or ſolid figures, there can no rectangle (which he im-
perly calls *a product*) be conceived to be made by any two
hem: And if it ſhould be ſaid, that in this caſe ſtraight lines
to be taken which are proportional to the figures, the de-
nſtration would this way become much longer than Eu-
's: But even, tho' his demonſtration had been right, who
s not ſee that it could not be made uſe of in the 5th Book?

PROP. F, G, H, K. B. V.

Theſe propoſitions are annexed to the 5th Book, becauſe they
frequently made uſe of by both antient and modern geome-
s: And in many caſes compound ratios cannot be brought
o demonſtrations, without making uſe of them.

Whoever deſires to ſee the doctrine of ratios delivered in
s 5th Book ſolidly defended, and the arguments brought a-
nſt it by And. Tacquet, Alph. Borellus, and others, fully
uted, may read Dr Barrow's mathematical lectures, viz. the
and 8th of the year 1666.

The 5th book being thus corrected, I moſt readily agree to
at the learned Dr Barrow ſays *, "That there is nothing * P. 336.
"in

Book V. "in the whole body of the Elements, of a more ſubtile inventi
"nothing more ſolidly eſtabliſhed and more accurately hand
"than the doctrine of proportionals." And there is ſ
ground to hope, that geometers will think that this could
have been ſaid with as good reaſon, ſince Theon's time till
preſent.

Book VI.

DEF. II. and V. of B. VI.

THE 2. definition does not ſeem to be Euclid's but ſ
unſkilful editor's: For there is no mention made by
clid, nor, as far as I know, by any other geometer, of reci
cal figures: It is obſcurely expreſſed, which made it pro
to render it more diſtinct: It would be better to put the follo
ing definition in place of it, viz.

DEF. II

Two magnitudes are ſaid to be reciprocally proportiona
two others, when one of the firſt is to one of the other m
nitudes, as the remaining one of the laſt two is to the remain
one of the firſt.

But the 5. definition, which, ſince Theon's time, has b
kept in the elements, to the great detriment of learners, is n
juſtly thrown out of them, for the reaſons given in the notes
the 23. prop. of this book.

PROP. I. and II. B. VI.

To the firſt of theſe a corollary is added which is often uſ
And the enunciation of the ſecond is made more general.

PROP. III. B. VI.

A ſecond caſe of this, as uſeful as the firſt, is given in pr
A, viz. the caſe in which the exterior angle of a triangle is
ſected by a ſtraight line: The demonſtration of it is very lik
that of the firſt caſe, and upon this account may, probably, h
been left out, as alſo the enunciation, by ſome unſkilful edi
At leaſt, it is certain, that Pappus makes uſe of this caſe, as
elementary propoſition, without a demonſtration of it, in pr
39. of his 7. book of Mathematical Collections.

PRO

PROP. VII. B. VI.

To this a caſe is added which occurs not unfrequently in demonſtrations.

PROP. VIII. B. VI.

t ſeems plain, that ſome editor has changed the demonion that Euclid gave of this propoſition: For, after he has monſtrated, that the triangles are equiangular to one another, particularly ſhews that their ſides about the equal angles are portionals, as if this had not been done in the demonſtration he 4. prop. of this book: This ſuperfluous part is not found in tranſlation from the Arabic, and is now left out.

PROP. IX. B. VI.

This is demonſtrated in a particular caſe, viz. that in which third part of a ſtraight line is required to be cut off; ich is not at all like Euclid's manner: Beſides, the author the demonſtration, from four magnitudes being proporals, concludes that the third of them is the ſame multiple he fourth, which the firſt is of the ſecond; now this is no re demonſtrated in the 5. book, as we now have it: But editor aſſumes it from the confuſed notion which the vulgar e of proportionals: On this account, it was neceſſary to give eneral and legitimate demonſtration of this propoſition.

PROP. XVIII. B. VI.

The demonſtration of this ſeems to be vitiated: For the poſition is demonſtrated only in the caſe of quadrilateral ures, without mentioning how it may be extended to figures ve or more ſides: Beſides, from two triangles being equiular, it is inferred, that a ſide of the one is to the homolos ſide of the other, as another ſide of the firſt is to the homologous to it of the other, without permutation of proportionals; which is contrary to Euclid's manner, as lear from the next propoſition: And the ſame fault occurs in in the concluſion, where the ſides about the equal angles not ſhewn to be proportionals; by reaſon of again neting permutation: On theſe accounts, a demonſtration is en in Euclid's manner, like to that he makes uſe of in the

20. prop.

Book VI. 20. Prop. of this book; and it is extended to five ſided figu by which it may be ſeen how to extend it to figures of any n ber of ſides.

PROP. XXIII. B. VI.

Nothing is uſually reckoned more difficult in the elem of geometry by learners, than the doctrine of compound tio, which Theon has rendered abſurd and ungeometrical, ſubſtituting the 5. definition of the 6. book in place of right definition, which without doubt Eudoxus or Euclid g in its proper place, after the definition of triplicate r &c. in the 5. book. Theon's definition is this; a ratio ſaid to be compounded of ratios ὁταν αἱ των λογων πλευρ εφ' ἑαυτας πολλαπλασιασθεισαι ποιωσι τινα: Which Comman thus tranſlates, "quando rationum quantitates inter ſe m "plicatae aliquam efficiunt rationem;" that is, when quantities of the ratios being multiplied by one another m a certain ratio. Dr Wallis tranſlates the word πλευρ "rationum exponentes," the exponents of the ratios: Dr Gregory renders the laſt words of the definition by "i "facit quantitatem," makes the quantity of that ratio: in whatever ſenſe the "quantities" or "exponents of the "tios," and their "multiplication" be taken, the defini will be ungeometrical and uſeleſs: For there can be no m plication but by a number; now the quantity or exponen a ratio (according as Eutocius in his Comment. on prop book 2. of Arch. de Sph. et Cyl. and the moderns explain term) is the number which multiplied into the conſequent t of a ratio produces the antecedent, or, which is the ſame th the number which ariſes by dividing the antecedent by conſequent; but there are many ratios ſuch, that no num can ariſe from the diviſion of the antecedent by the conſeque ex. gr. the ratio which the diameter of a ſquare has to ſide of it; and the ratio which the circumference of a c has to its diameter, and ſuch like. Beſides, there is not leaſt mention made of this definition in the writings of clid, Archimedes, Apollonius, or other antients, tho' they quently make uſe of compound ratio: And in this 23 P of the 6. book, where compound ratio is firſt mentio there is not one word which can relate to this definition, here, if in any place, it was neceſſary to be brought in; but right definition is expreſsly cited in theſe words: "But "ratio of K to M is compounded of the ratio of K to

and of the ratio of L to M." This definition therefore of Theon is quite uſeleſs and abſurd: For that Theon brought it to the elements can ſcarce be doubted; as it is to be found in his commentary upon Ptolomy's Μεγαλη Συνταξις, page 62. where he alſo gives a childiſh explication of it, as agreeing only to ſuch ratios as can be expreſſed by numbers; and from this place the definition and explication have been exactly copied and prefixed to the definitions of the 6th book, as appears from Hervagius's edition: But Zambertus and Commandine, in their Latin tranſlations, ſubjoin the ſame to theſe definitions. Neither Campanus, nor, as it ſeems, the Arabic manuſcripts, from which he made his tranſlation, have this definition Clavius, in his obſervations upon it, rightly judges that the definition of compound ratio might have been made after the ſame manner in which the definitions of duplicate and triplicate ratio are given, viz. "That as in ſeveral magnitudes that are continual proportionals, Euclid named the ratio of the firſt to the third, the duplicate ratio of the firſt to the ſecond; and the ratio of the firſt to the fourth, the triplicate ratio of the firſt to the ſecond; that is, the ratio compounded of two or three intermediate ratios that are equal to one another, and ſo on; ſo, in like manner, if there be ſeveral magnitudes of the ſame kind, following one another, which are not continual proportionals, the firſt is ſaid to have to the laſt the ratio compounded of all the intermediate ratios,——only for this reaſon, that theſe intermediate ratios are interpoſed betwixt the two extremes, viz. the firſt and laſt magnitudes; even as, in the 10th definition of the 5th book, the ratio of the firſt to the third was called the duplicate ratio, merely upon account of two ratios being interpoſed betwixt the extremes, that are equal to one another: So that there is no difference betwixt this compounding of ratios, and the duplication or triplication of them which are defined in the 5th book, but that in the duplication, triplication, &c. of ratios, all the interpoſed ratios are equal to one another; whereas, in the compounding of ratios, it is not neceſſary that the intermediate ratios ſhould be equal to one another." Alſo Mr Edmund Scarburgh, in his Engliſh tranſlation of the firſt ſix books, page 238. 6 expreſsly affirms, that the 5th definition of the 6th book, is ſuppoſitious, and that the true definition of compound ratio

Book VI. 20. Prop. of this book; and it is extended to five sided figu by which it may be ſeen how to extend it to figures of any n ber of ſides.

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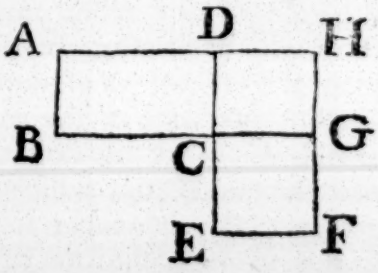
Book VI. is contained in the 10th definition of the 5th book, viz. th definition of duplicate ratio, or to be underſtood from it, t wit, in the ſame manner as Clavius has explained it in the pre ceeding citation. Yet theſe, and the reſt of the moderns, notwithſtanding retain this 5th def. of the 6th Book, and illuſtra and explain it by long commentaries, when they ought rath to have taken it quite away from the elements.

For, by comparing def. 5th, Book 6th, with prop. 5th, Boo 8. it will clearly appear that this definition has been put into th elements in place of the right one which has been taken o of them: Becauſe, in prop. 5. Book 8. it is demonſtrated th the plane number of which the ſides are C, D has to the pla number of which the ſides are E, Z (ſee Hervagius's Gregory's edition), the ratio which is compounded of the r tios of their ſides; that is, of the ratios of C to E, and D Z: And by def. 5. Book 6. and the explication given of it all the commentators, the ratio which is compounded of the r tios of C to E, and D to Z, is the ratio of the product ma by the multiplication of the antecedents C, D to the produ of the conſequents E, Z; that is, the ratio of the plane numb of which the ſides are C, D to the plane number of wh the ſides are E, Z. Wherefore the propoſition which is the 5 def. of Book 6. is the very ſame with the 5th prop. of Book and therefore it ought neceſſarily to be cancelled in one of th places; becauſe it is abſurd that the ſame propoſition ſho ſtand as a definition in one place of the elements, and be monſtrated in another place of them. Now there is no do that prop. 5. Book 8. ſhould have a place in the elements, the ſame thing is demonſtrated in it concerning plane nu bers, which is demonſtrated in prop. 23. Book 6. of equian lar parallelograms; wherefore def. 5. Book 6. ought not to in the elements. And from this it is evident that this definit is not Euclid's, but Theon's, or ſome other unſkilful geomete

But no body, as far as I know, has hitherto ſhewn the t uſe of compound ratio, or for what purpoſe it has been troduced into geometry; for every propoſition in wh compound ratio is made uſe of, may without it be both en ciated and demonſtrated. Now the uſe of compound r conſiſts wholly in this, that by means of it, circumlocuti may be avoided, and thereby propoſitions may be more b ly either enunciated or demonſtrated, or both may be d for inſtance, if this 23d propoſition of the 6th book were be enunciated, without mentioning compound ratio, it m

e done as follows. If two parallelograms be equiangular, and as a ſide of the firſt to a ſide of the ſecond, ſo any aſſumed traight line be made to a ſecond ſtraight line; and as the o-her ſide of the firſt to the other ſide of the ſecond, ſo the ſe-ond ſtraight line be made to a third. The firſt parallelogram to the ſecond, as the firſt ſtraight line to the third. And the emonſtration would be exactly the ſame as we now have it. ut the antient geometers, when they obſerved this enuncia-on could be made ſhorter, by giving a name to the ratio hich the firſt ſtraight line has to the laſt, by which name the ntermediate ratios might likewiſe be ſignified, of the firſt to e ſecond, and of the ſecond to the third, and ſo on, if there ere more of them, they called this ratio of the firſt to the ſt, the ratio compounded of the ratios of the firſt to the ſe-ond, and of the ſecond to the third ſtraight line; that is, in e preſent example, of the ratios which are the ſame with e ratios of the ſides, and by this they expreſſed the propo-ion more briefly thus: If there be two equiangular paralle-grams, they have to one another the ratio which is the me with that which is compounded of ratios that are the me with the ratios of the ſides. Which is ſhorter than the receeding enunciation, but has preciſely the ſame meaning. r yet ſhorter thus: Equiangular parallelograms have to one other the ratio which is the ſame with that which is com-unded of the ratios of their ſides. And theſe two enuncia-ns, the firſt eſpecially, agree to the demonſtration which is w in the Greek. The propoſition may be more briefly de-onſtrated, as Candalla does, thus: Let ABCD, CEFG be o equiangular parallelograms, and complete the paralielo-am CDHG; then, becauſe there are three parallelograms C, CH, CF, the firſt AC (by the definition of compound io) has to the third CF, the ratio ich is compounded of the ratio of e firſt AC to the ſecond CH, and of e ratio of CH to the third CF; but parallelogram AC is to the pa-elogram CH, as the ſtraight line to CG; and the parallelogram H is to CF, as the ſtraight line D is to CE; therefore the parallelogram AC has to CF the o which is compounded of ratios that are the ſame with the os of the ſides. And to this demonſtration agrees the enun-ion which is at preſent in the text, viz. Equiangular paralle-rams have to one another the ratio which is compounded of



Book VI. the ratios of the ſides: For the vulgar reading, " which is com-
" pounded of their ſides," is abſurd. But, in this edition we have kept the demonſtration which is in the Greek text, though not ſo ſhort as Candalla's; becauſe the way of finding the ratio which is compounded of the ratios of the ſides, that is, of finding the ratio of the parallelograms, is ſhewn in that, but not in Candalla's demonſtration; whereby beginners may learn, in like caſes, how to find the ratio which is compounded of two or more given ratios.

From what has been ſaid, it may be obſerved, that in any magnitudes whatever of the ſame kind A, B, C, D, &c. the ratio compounded of the ratios of the firſt to the ſecond, of the ſecond to the third, and ſo on to the laſt, is only a name or expreſſion by which the ratio which the firſt A has to the laſt D is ſignified, and by which at the ſame time the ratios of all the magnitudes A to B, B to C, C to D from the firſt to the laſt, to one another, whether they be the ſame, or be not the ſame, are indicated; as, in magnitudes which are continual proportionals A, B, C, D, &c. the duplicate ratio of the firſt to the ſecond is only a name, or expreſſion by which the ratio of the firſt A to the third C is ſignified, and by which, at the ſame time, is ſhewn that there are two ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of the ſecond B to the third or laſt C, which are the ſame with one another; and the triplicate ratio of the firſt to the ſecond is a name or expreſſion by which the ratio of the firſt A to the fourth D is ſignified, and by which, at the ſame time, is ſhewn that there are three ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of B to the third C, and of C to the fourth or laſt D, which are all the ſame with one another; and ſo in the caſe of any other multiplicate ratios. And that this is the right explication of the meaning of theſe ratios is plain from the definitions of duplicate and triplicate ratio in which Euclid makes uſe of the word λέγεται, is ſaid to be, or is called; which word, he no doubt made uſe of alſo in the definition of compound ratio, which Theon, or ſome other, has expunged from the elements; for the very ſame word is ſtill retained in the wrong definition of compound ratio which is now the 5th of the 6th book: But in the citation of theſe definitions it is ſometimes retained, as in the demonſtration of prop. 1 Book

ook 6. "the firſt is ſaid to have, ἔχειν λέγεται, to the third the duplicate ratio," &c. which is wrong tranſlated by Commandine and others, "has" inſtead of "is ſaid to have;" and ometimes it is left out, as in the demonſtration of prop. 33. f the 11th Book, in which we find "the firſt has, ἔχει, to the third e triplicate ratio;" but without doubt ἔχει, "has," in this lace ſignifies the ſame as ἔχειν λέγεται, is ſaid to have: So likeiſe in prop. 23. B. 6. we find this citation, "but the ratio of K to M is compounded, σύγκειται, of the ratio of K to L, and the ratio of L to M," which is a ſhorter way of expreſsng the ſame thing, which, according to the definition, ought have been expreſſed by συγκεῖσθαι λέγεται, is ſaid to be compounded.

From theſe remarks, together with the propoſitions ſubjoined the 5th book, all that is found concerning compound ratio ther in the ancient or modern geometers may be underſtood nd explained.

PROP. XXIV. B. VI.

It ſeems that ſome unſkilful editor has made up this demonration as we now have it, out of two others; one of which may e made from the 2d prop. and the other from the 4th of this ook: For after he has, from the 2d of this book, and compoſitin and permutation, demonſtrated that the ſides about the angle ommon to the two parallelograms are proportionals, he might ave immediately concluded that the ſides about the other equal ngles were proportionals, viz. from prop. 34. B. 1. and prop. Book 5. This he does not, but proceeds to ſhew that the triangles and parallelograms are equiangular; and in a tedious ay, by help of prop. 4. of this book, and the 22d of Book 5. educes the ſame concluſion: From which it is plain that this compoſed demonſtration is not Euclid's: Theſe ſuperfluous ings are now left out, and a more ſimple demonſtration is iven from the 4th prop. of this book, the ſame which is in the anſlation from the Arabic, by help of the 2d prop. and compoſition; but in this the author neglects permutation, and does ot ſhew the parallelograms to be equiangular, as is proper to o for the ſake of beginners.

PROP. XXV. B. VI.

It is very evident that the demonſtration which Euclid had iven of this propoſition has been vitiated by ſome unſkilful and: For, after this editor had demonſtrated that "as the rectilineal figure ABC is to the rectilineal KGH, ſo is the

Book VI. " parallelogram BE to the parallelogram EF;" nothing more ſhould have been added but this, " and the rectilineal figure " ABC is equal to the parallelogram BE; therefore the rectilineal KGH is equal to the parallelogram EF," viz. from prop. 14. Book 5. But betwixt theſe two ſentences he has inſerted this; " wherefore, by permutation, as the rectilineal figure ABC to the parallelogram BE, ſo is the rectilineal KGH " to the parallelogram EF;" by which, it is plain, he thought it was not ſo evident to conclude that the ſecond of four proportionals is equal to the fourth from the equality of the firſt and third, which is a thing demonſtrated in the 14th prop. of B. 5. as to conclude that the third is equal to the fourth, from the equality of the firſt and ſecond, which is no where demonſtrated in the elements as we now have them: But, tho' this propoſition, viz. the third of four proportionals is equal to the fourth, if the firſt be equal to the ſecond, had been given in the elements by Euclid, as very probably it was, yet he would not have made uſe of it in this place; becauſe, as was ſaid, the concluſion could have been immediately deduced without this ſuperfluous ſtep by permutation: This we have ſhewn at the greater length, both becauſe it affords a certain proof of the vitiation of the text of Euclid, for the very ſame blunder is found twice in the Greek text of prop. 23. Book 11. and twice in prop. 2. B. 12. and in the 5. 11. 12. and 18th of that book in which places of Book 12. except the laſt of them, it is rightly left out in the Oxford edition of Commandine's tranſlation. And alſo that geometers may beware of making uſe of permutation in the like caſes, for the moderns not unfrequently commit this miſtake, and among others Commandine himſelf in his commentary on prop. 5 Book 3. p. 6. b. of Pappus Alexandrinus and in other places: The vulgar notion of proportionals has, it ſeems, preoccupied many ſo much, that they do not ſufficiently underſtand the true nature of them.

Beſides, though the rectilineal figure ABC, to which another is to be made ſimilar, may be of any kind whatever, yet in the demonſtration the Greek text has " triangle" inſtead of rectilineal figure,' which error is corrected in the above named Oxford edition.

PROP. XXVII. B. VI.

The ſecond caſe of this has ἀλλως, otherwiſe, prefixed to it, as if it was a different demonſtration, which probably has been done by ſome unſkilful Librarian. Dr Gregory has right

y left it out: The ſcheme of this ſecond caſe ought to be mark-
d with the ſame letters of the alphabet which are in the ſcheme
f the firſt, as is now done.

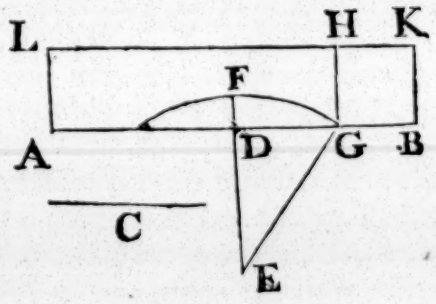
PROP. XXVIII. and XXIX. B. VI.

Theſe two problems, to the firſt of which the 27th prop. is neceſſary, are the moſt general and uſeful of all in the elements, and are moſt frequently made uſe of by the ancient geometers in the ſolution of other problems; and therefore are very ignorantly left out by Tacquet and Dechales in their editions of the Elements, who pretend that they are ſcarce of any uſe: The caſes of theſe problems, wherein it is required to apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, either deficient or exceeding by a ſquare; as alſo to apply a rectangle which ſhall be equal to another given, to a given ſtraight line, deficient or exceeding by a ſquare, are very often made uſe of by geometers: And, on this account, it is thought proper, for the ſake of beginners, to give their conſtructions, as follows.

1. To apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, deficient by a ſquare: But the given ſquare muſt not be greater than that upon the half of the given line.

Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be that to which the rectangle to be applied muſt be equal, and this ſquare, by the determination, is not greater than that upon half of the ſtraight line AB.

Biſect AB in D, and if the ſquare upon AD be equal to the ſquare upon C, the thing required is done: But if it be not equal to it, AD muſt be greater than C, according to the determination: Draw DE at right angles to AB, and make it equal to C; produce ED to F, ſo that EF be equal to AD or DB, and from the center E, at the diſtance EF, deſcribe a circle meeting AB in G, and upon GB deſcribe the ſquare GBKH, and complete the rectangle AGHL; alſo join EG: And becauſe AB is biſected in D, the rectangle AG, GB together with the ſquare of DG is equal [a] to (the ſquare of DB, that is, of EF or EG, that is, to)



[a] 5. 2.

Book VI.

to) the ſquares of ED, DC: Take away the ſquare of D from each of theſe equals; therefore the remaining rectang AG, GB is equal to the ſquare of ED, that is, of C: But t rectangle AG, GB is the rectangle AH, becauſe GH is equ to GB: Therefore the rectangle AH is equal to the given ſqua upon the ſtraight line C. Wherefore the rectangle AH, equ to the given ſquare upon C, has been applied to the giv ſtraight line AB, deficient by the ſquare GK. Which was be done.

2. To apply a rectangle which ſhall be equal to a given ſqua to a given ſtraight line, exceeding by a ſquare.

Let AB be the given ſtraight line, and let the ſquare up the given ſtraight line C be that to which the rectangle to applied muſt be equal.

Biſect AB in D, and draw BE at right angles to it, ſo th BE be equal to C; and having joined DE, from the center at the diſtance DE deſcribe a circle meeting AB produced G; upon BG deſcribe the ſquare BGHK, and complete the rectangle AGHL. And becauſe AB is biſected in D, and produced to G, the rectangle AG, GB together with the ſquare of

[a] 6. 2.

DB is equal [a] to (the ſquare of DG, or DE, that is, to) the ſquares of EB, BD. From each of theſe equals take the ſquare of DB; therefore the remaining rectangle AG, GB is equal to t ſquare of BE, that is, to the ſquare upon C. But the rectang AG, GB is the rectangle AH, becauſe GH is equal to G Therefore the rectangle AH is equal to the ſquare upon Wherefore the rectangle AH, equal to the given ſquare up C, has been applied to the given ſtraight line AB, exceedi by the ſquare GK. Which was to be done.

3. To apply a rectangle to a given ſtraight line which ſh be equal to a given rectangle, and be deficient by a ſquare. B the given rectangle muſt not be greater than the ſquare up the half of the given ſtraight line.

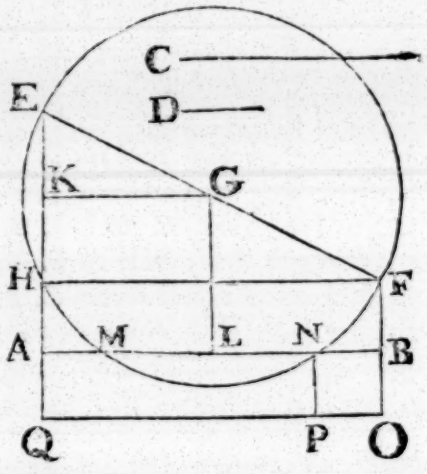
Let AB be the given ſtraight line, and let the given rectang be that which is contained by the ſtraight lines C, D, which not greater than the ſquare upon the half of AB; it is requir to apply to AB a rectangle equal to the rectangle C, D, de cient by a ſquare.

Dra

Book VI.

Draw AE, BF at right angles to AB, upon the ſame ſide of it, and make AE equal to C, and BF to D: Join EF and biſect it in G; and from the center G, at the diſtance GE, deſcribe a circle meeting AE again in H; join HF and draw GK parallel to it, and GL parallel to AE meeting AB in L.

Becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF are parallels, and AH and BF are parallels; wherefore AH is equal to BF, and the rectangle EA, AH equal to the rectangle EA, BF, that is to the rectangle C, D: And becauſe EG, GF are equal to one another, and AE, LG, BF parallels, therefore AL and LB are equal; alſo EK is equal to KH [a], and the rectangle C, D, from the determination, is not greater than the ſquare of AL the half of AB; wherefore the rectangle EA, AH is not greater than the ſquare of AL, that is of KG: Add to each the ſquare of KE; therefore the ſquare [b] of AK is not greater than the ſquares of EK, KG, that is, than the ſquare of EG; and conſequently the ſtraight line AK or GL is not greater than GE. Now, if GE be equal to GL, the circle EHF touches AB in L, and therefore the ſquare of AL is [c] equal to the rectangle EA, AH, that is, to the given rectangle C, D; and that which was required is done: But if EG, GL be unequal, EG muſt be the greater; and therefore the circle EHF cuts the ſtraight line AB; let it cut it in the points M, N, and upon NB deſcribe the ſquare NBOP, and complete the rectangle ANPQ: Becauſe ML is equal to [d] LN, and it has been proved that AL is equal to LB; therefore AM is equal to NB, and the rectangle AN, NB equal to the rectangle NA, AM, that is, to the rectangle [e] EA, AH or the rectangle C, D: But the rectangle AN, NB is the rectangle AP, becauſe PN is equal to NB: Therefore the rectangle AP is equal to the rectangle C, D; and the rectangle A, P equal to the given rectangle C, D has been applied to the given ſtraight line AB, deficient by the ſquare BP. Which was to be done.



[a] 3. 3.

[b] 6. 2.

[c] 36. 3.

[d] 3. 3.

[e] Cor. 36. 3.

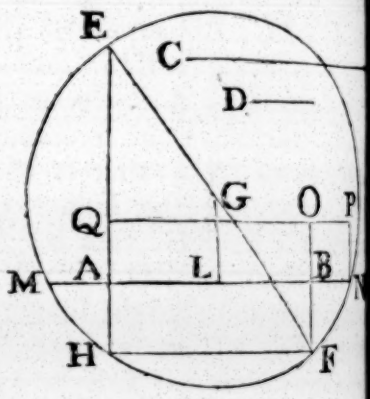
4. To

Book VI.

4. To apply a rectangle to a given ſtraight line that ſhall equal to a given rectangle, exceeding by a ſquare.

Let AB be the given ſtraight line, and the rectangle C, the given rectangle, it is required to apply a rectangle to A equal to C, D, exceeding by a ſquare.

Draw AE, BF at right angles to AB, on the contr ſides of it, and make AE equal to C, and BF equal to Join EF, and biſect it in G; and from the center G, at diſtance GE, deſcribe a circle meeting AE again in H; j HF, and draw GL parallel to AE; let the circle meet AB produced in M, N, and upon BN deſcribe the ſquare NBOP, and complete the rectangle ANPQ: becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF are parallels, and therefore AH and BF are equal, and the rectangle EA, AH equal to the rectangle EA, BF, that is, to the rectangle C, D: And becauſe ML equal to LN, and AL to LB, therefore MA is equal to BN, a the rectangle AN, NB to MA, AN, that is, [a] to the rectan EA, AH, or the rectangle C, D: Therefore the rectangle A NB, that is, AP, is equal to the rectangle C, D; and to the ven ſtraight line AB the rectangle AP has been applied equa the given rectangle C, D, exceeding by the ſquare BP. Wh was to be done.



a 35. 3.

Willebrordus Snellius was the firſt, as far as I know, gave theſe conſtructions of the 3d and 4th problems in his pollonius Batavus: And afterwards the learned Dr Halley them in the Scholium of the 18th prop. of the 8th B. of Ap nius's conics reſtored by him.

The 3d problem is otherwiſe enunciated thus: To cu given ſtraight line AB in the point N, ſo as to make the angle AN, NB equal to a given ſpace: Or, which is ſame thing, having given AB the ſum of the ſides of a angle, and the magnitude of it being likewiſe given, to find ſides.

And the 4th problem is the ſame with this, To find a

n the given ſtraight line AB produced, ſo as to make the angle AN, NB equal to a given ſpace: Or, which is the e thing, having given AB the difference of the ſides of a rect-le, and the magnitude of it, to find the ſides.

PROP. XXXI. B. VI.

n the demonſtration of this, the inverſion of proportionals is e neglected, and is now added, that the concluſion may be imately made by help of the 24th prop. of B. 5. as Clavius done.

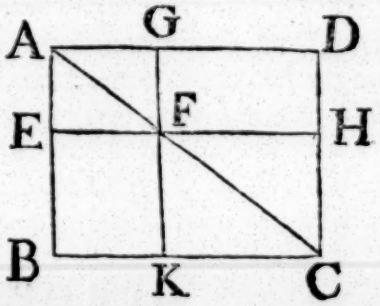
PROP. XXXII. B. VI.

The enunciation of the preceeding 26th prop. is not general e-gh; becauſe not only two ſimilar parallelograms that have ngle common to both, are about the ſame diameter; but like-two ſimilar parallelograms that have vertically oppoſite an-, have their diameters in the ſame ſtraight line: But there s to have been another, and that a direct demonſtration of e caſes, to which this 32d propoſition was needful: And the may be otherwiſe and ſomething more briefly demonſtrated ollows.

PROP. XXXII. B. VI.

f two triangles which have two ſides of the one, &c.

Let GAF, HFC be two triangles which have two ſides AG, , proportional to the two ſides FH, HC, viz. AG to GF, as to HC; and let AG be paral-to FH, and GF to HC; AF FC are in a ſtraight line.



Draw CK parallel [a] to FH, and it meet GF produced in K: auſe AG, KC are each of them allel to FH, they are parallel [b] one another, and therefore the rnate angles AGF, FKC are e-l: And AG is to GF, as (FH to HC, that is [c]) CK to KF; erefore the triangles AGF, CKF are equiangular [d], and the le AFG equal to the angle CFK: But GFK is a ſtraight line, erefore AF and FC are in a ſtraight line [e].

[a] 31. 1.

[b] 30. 1.

[c] 34. 1.

[d] 6. 6.

[e] 14. 1.

The 26th prop. is demonſtrated from the 32d, as follows.

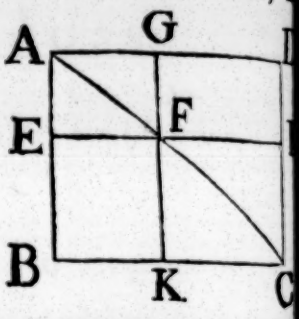
f two ſimilar and ſimilarly placed parallelograms have an an-common to both, or vertically oppoſite angles; their diame-are in the ſame ſtraight line.

First,

Book VI.

First, Let the parallelograms ABCD, AEFG have the an BAD common to both, and be ſimilar, and ſimilarly place ABCD, AEFG are about the ſame diameter.

Produce EF, GF, to H, K, and join FA, FC: Then cauſe the parallelograms ABCD, AEFG are ſimilar, I is to AB, as GA to AE; where-



[a] Cor. 19. 5. fore the remainder DG is [a] to the remainder EB, as GA to AE: But DG is equal to FH, EB to HC, and AE to GF: Therefore as FH to HC, ſo is AG to GF; and FH, HC are parallel to AG, GF; and the triangles AGF, FHC are joined at one angle, in the point

[b] 32. 6. F; wherefore AF, FC are in the ſame ſtraight line [b].

Next, Let the parallelograms KFHC, GFEA, which are ſi lar and ſimilarly placed, have their angles KFH, GFE vertic oppoſite; their diameters AF, FC, are in the ſame ſtraight l

Becauſe AG, GF are parallel to FH, HC; and that A to GF, as FH to HC; therefore AF, FC are in the ſame ſtra line [b].

PROP XXXIII. B. VI.

The words "becauſe they are at the center," are left ou the addition of ſome unſkilful hand.

In the Greek, as alſo in the Latin tranſlation, the w ἀ ετυχε, "any whatever," are left out in the demonſtrati both parts of the propoſition, and are now added as quite n ſary; and, in the demonſtration of the ſecond part, where th angle BGC is proved to be equal to CGK, the illative par ἄρα in the Greek text ought to be omitted.

The ſecond part of the propoſition is an addition of The as he tells us in his commentary on Ptolomy's Μεγάλη Συν p. 50.

PROP. B. C. D. B. VI.

Theſe three propoſitions are added, becauſe they are frequ ly made uſe of by geometers.

D

DEF. IX. and XI. B. XI.

THE ſimilitude of plane figures is defined from the equality of their angles, and the proportionality of the ſides about the equal angles; for from the proportionality of the ſides only, or only from the equality of the angles, the ſimilitude of the figures does not follow, except in the caſe when the figures are triangles: The ſimilar poſition of the ſides, which contain the figures, to one another, depending partly on each of theſe: And, for the ſame reaſon, thoſe are ſimilar ſolid figures which have all their ſolid angles equal, each to each, and are contained by the ſame number of ſimilar plane figures: For there are ſome ſolid figures contained by ſimilar plane figures, of the ſame number, and even of the ſame magnitude, that are neither ſimilar nor equal, as ſhall be demonſtrated after the notes on the 10th definition: Upon this account it was neceſſary to amend the definition of ſimilar ſolid figures, and to place the definition of a ſolid angle before it: And from this and the 10th definition, it is ſufficiently plain, how much the elements have been ſpoiled by unſkilful editors.

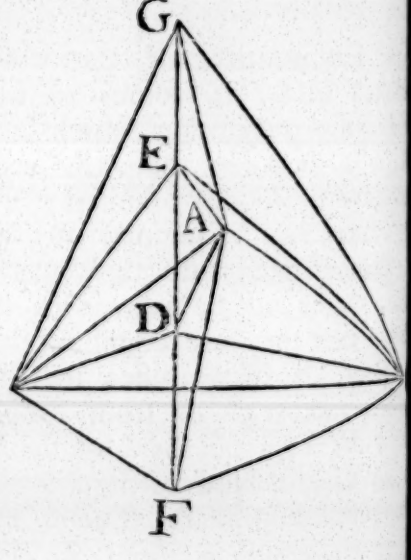
DEF. X. B. XI.

Since the meaning of the word "equal" is known and eſtabliſhed before it comes to be uſed in this definition; therefore the propoſition which is the 10th definition of this book, is a theorem, the truth or falſehood of which ought to be demonſtrated, not aſſumed; ſo that Theon, or ſome other Editor, has ignorantly turned a theorem which ought to be demonſtrated into this 10th definition: That figures are ſimilar, ought to be proved from the definition of ſimilar figures; that they are equal ought to be demonſtrated from the axiom, "Magnitudes that wholly coincide, are equal to one another;" or from Prop. A. of Book 5 or the 9th Prop. or the 14th of the ſame book, from one of which the equality of all kind of figures muſt ultimately be deduced. In the preceeding books, Euclid has given no definition of equal figures, and it is certain he did not give this: For what is called the 1ſt def. of the 3d book, is really a theorem in which theſe circles are ſaid to be equal, that have the ſtraight lines from their centers to the circumferences equal, which is plain, from the definition of a circle; and therefore has by

ſome

Book XI. ſome editor been improperly placed among the definitions. T
equality of figures ought not to be defined, but demonſtra
Therefore, though it were true, that ſolid figures contain
the ſame number of ſimilar and equal plane figures are equ
one another, yet he would juſtly deſerve to be blamed who ſh
make a definition of this propoſition which ought to be dem
ſtrated. But if this propoſition be not true, muſt it not be c
eſſed, that geometers have, for theſe thirteen hundred years, b
miſtaken in this elementary matter? And this ſhould teach
modeſty, and to acknowledge how little, through the weak
of our minds, we are able to prevent miſtakes even in the p
ciples of ſciences which are juſtly reckoned amongſt the m
certain; for that the propoſition is not univerſally true, ca
ſhewn by many examples: The following is ſufficient.

Let there be any plane rectilineal figure, as the trian
[a] 12. 11. ABC, and from a point D within it draw [a] the ſtraight
DE at right angles to the plane ABC; in DE take DE, D
equal to one another, upon the oppoſite ſides of the pl
and let G be any point in EF; join DA, DB, DC; E
EB, EC; FA, FB, FC; GA, GB, GC: Becauſe the ſtra
line EDF is at right angles to the plane ABC, it makes r
angles with DA, DB, DC which it meets in that plane;
in the triangles EDB, FDB, ED and DB are equal to FD
DB, each to each, and they contain right angles; there
[b] 4. 1. the baſe EB is equal [b] to the baſe FB; in the ſame manner EA is equal to FA, and EC to FC: And in the triangles EBA, FBA, EB, BA are equal to FB, BA and the baſe EA is equal to the baſe FA; wherefore the angle
[c] 8. 2. EBA is equal [c] to the angle FBA, and the triangle EBA equal [b] to the triangle FBA, and the other angles equal to
[d] 4. 6. 1. def. 6. the other angles; therefore theſe triangles are ſimilar [d]: In the ſame manner the triangle EBC is ſimila



triangle FBC, and the triangle EAC to FAC; therefore Book XI.
e are two ſolid figures each of which is contained by ſix
ngles, one of them by three triangles, the common vertex
hich is the point G, and their baſes the ſtraight lines AB,
, CA; and by three other triangles the common vertex
hich is the point E, and their baſes the ſame lines AB,
, CA: The other ſolid is contained by the ſame three tri-
les the common vertex of which is G, and their baſes AB,
, CA; and by three other triangles of which the common
ex is the point F, and their baſes the ſame ſtraight lines
, BC, CA: Now the three triangles GAB, GBC, GCA
common to both ſolids, and the three others EAB, EBC,
A of the firſt ſolid have been ſhewn equal and ſimilar to the
e others FAB, FBC, FCA of the other ſolid, each to each;
refore theſe two ſolids are contained by the ſame number of
al and ſimilar planes: But that they are not equal is mani-
, becauſe the firſt of them is contained in the other: There-
it is not univerſally true that ſolids are equal which are
tained by the ſame number of equal and ſimilar planes.

Cor. From this it appears that two unequal ſolid angles
y be contained by the ſame number of equal plane angles.
For the ſolid angle at B which is contained by the four plain
les EBA, EBC, GBA, GBC is not equal to the ſolid an-
at the ſame point B which is contained by the four plane
les FBA, FBC, GBA, GBC; for this laſt contains the o-
r: And each of them is contained by four plane angles,
ich are equal to one another, each to each, or are the ſelf
e; as has been proved: And, indeed, there may be innu-
rable ſolid angles all unequal to one another, which are each
them contained by plane angles that are equal to one ano-
r, each to each: It is likewiſe manifeſt that the before
ntioned ſolids are not ſimilar, ſince their ſolid angles are not
equal.

And that there may be innumerable ſolid angles all unequal
one another, which are each of them contained by the
e plane angles diſpoſed in the ſame order, will be plain
m the three following propoſitions.

PROP. I. PROBLEM.

Three magnitudes A, B, C being given, to find a fourth
h, that every three ſhall be greater than the remaining one.
Let D be the fourth; therefore D muſt be leſs than A, B,
C together:

Book XI.

C together: Of the three A, B, C, let A be that which is leſs than either of the two B and C: And firſt, let B and together be not leſs than A: therefore B, C, D toge are greater than A; and becauſe A is not leſs than B, C, D together are greater than B: In the like manner B, D together are greater than C: Wherefore in the caſe which B and C together are not leſs than A, any magnit D which is leſs than A, B, C together will anſwer the p blem.

But if B and C together be leſs than A; then, becauſe i required that B, C, D together be greater than A, from e of theſe taking away B, C, the remaining one D muſt greater than the exceſs of A above B and C: Take there any magnitude D which is leſs than A, B, C together, greater than the exceſs of A above B and C: Then B, C together are greater than A; and becauſe A is greater t either B or C, much more will A and D, together with ei of the two B, C be greater than the other: And, by the c ſtruction, A, B, C are together greater than D.

Cor. If beſides it be required, that A and B toge ſhall not be leſs than C and D together; the exceſs of A B together above C muſt not be leſs than D, that is, D n not be greater than that exceſs.

PROP. II. PROBLEM.

Four magnitudes A, B, C, D being given, of which A B together are not leſs than C and D together, and ſuch any three of them whatever are greater than the fourth is required to find a fifth magnitude E ſuch, that any tw the three A, B, E ſhall be greater than the third, and that any two of the three C, D, E ſhall be greater than third. Let A be not leſs than B: And C not leſs than D.

Firſt, Let the exceſs of C above D be not leſs than the ceſs of A above B: It is plain that a magnitude E can taken which is leſs than the ſum of C and D, but gre than the exceſs of C above D; let it be taken; then greater likewiſe than the exceſs of A above B; wherefor and B together are greater than A; and A is not leſs than therefore A and E together are greater than B: And, by hypotheſis, A and B together are not leſs than C and D t ther, and C and D together are greater than E; therefore wiſe A and B are greater than E.

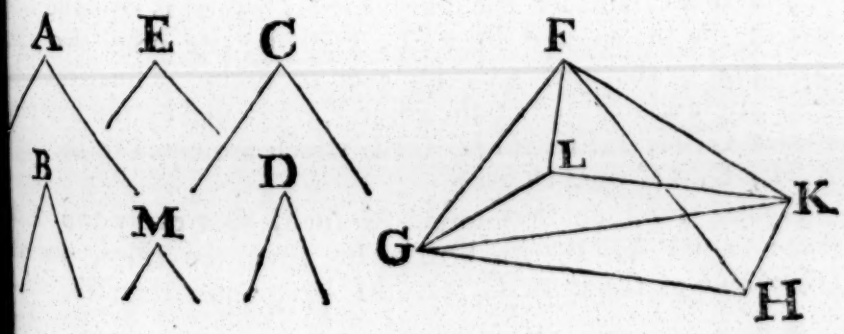
But let the exceſs of A above B be greater than the exceſs C above D: And becauſe, by the hypotheſis, the three B, D are together greater than the fourth A; C and D toge-er are greater than the exceſs of A above B: Therefore a agnitude may be taken which is leſs than C and D together, t greater than the exceſs of A above B. Let this magnitude E; and becauſe E is greater than the exceſs of A above B, together with E is greater than A: And, as in the preceeding ſe, it may be ſhewn that A together with E is greater than and that A together with B is greater than E: Therefore, in ch of the caſes, it has been ſhewn that any two of the three B, E are greater than the third.

And becauſe in each of the caſes E is greater than the ex-ſs of C above D, E together with D is greater than C; and, the hypotheſis, C is not leſs than D; therefore E together th C is greater than D; and, by the conſtruction, C and D gether are greater than E: Therefore any two of the three, D, E are greater than the third.

PROP. III. THEOREM.

There may be innumerable ſolid angles all unequal to one other, each of which is contained by the ſame four plane an-es, placed in the ſame order.

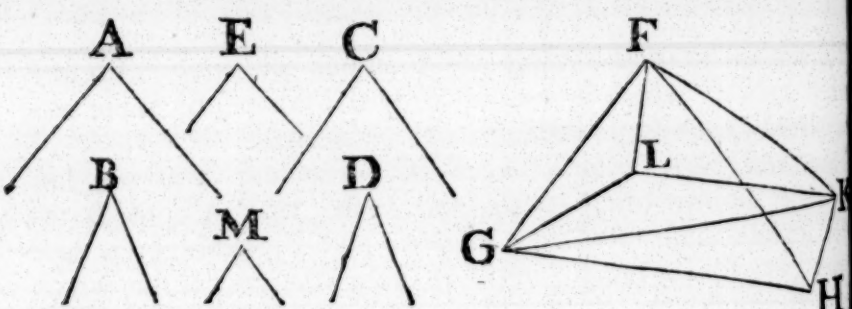
Take three plane angles, A, B, C, of which A is not leſs n either of the other two, and ſuch, that A and B to-ther are leſs than two right angles; and by problem 1. and corollary, find a fourth angle D ſuch, that any three what-r of the angles A, B, C, D be greater than the remaining gle, and ſuch, that A and B together be not leſs than C d D together: And by problem 2. find a fifth angle E ſuch, t any two of the angles A, B, E be greater than the third.



d alſo that any two of the angles C, D, E be greater than

 the

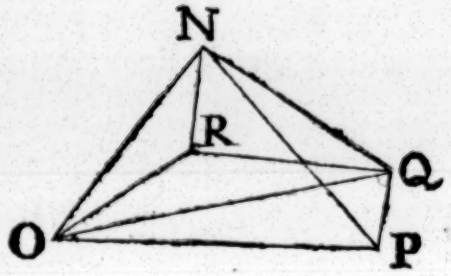
book XI. the third: And becaule A and B together are leſs than tw
right angles, the double of A and B together is leſs than fo
right angles: But A and B together are greater than the ang
E; wherefore the double of A and B together is greater tha
the three angles A, B, E together, which three are conſe
quently leſs than four right angles; and every two of th
ſame angles A, B, E are greater than the third; therefor
by prop 23. 11. a ſolid angle may be made contained by thre
plane angles equal to the angles A, B, E, each to each. L
this be the angle F contained by the three plane angles GF
HFK, GFK which are equal to the angles A, B, E, each t
each: And becauſe the angles C, D together are not greate
than the angles A, B together, therefore the angles C, D,
are not greater than the angles A, B, E: But theſe laſt three a
leſs than four right angles, as has been demonſtrated; wher
fore alſo the angles C, D, E are together leſs than four rig
angles, and every two of them are greater than the third; ther
fore a ſolid angle may be made which ſhall be contained by thr
[a] 23. 11. plane angles equal to the angles C, D, E, each to each [a]: An



by prop. 26. 11. at the point F in the ſtraight line FG a ſol
angle may be made equal to that which is contained by t
three plane angles that are equal to the angles C, D, E: L
this be made, and let the angle GFK, which is equal to E,
one of the three; and let KFL, GFL be the other two whi
are equal to the angles C, D, each to each. Thus, there is
ſolid angle conſtituted at the point F contained by the four pla
angles GFH, HFK, KFL, GFL which are equal to the angl
A, B, C, D, each to each.

Again, Find another angle M ſuch, that every two of t
three angles A, B, M be greater than the third, and a
every two of the three C, D, M be greater than the thir
An

nd, as in the preceeding part, it may be demonſtrated that Book XI.
e three A, B, M are leſs
an four right angles, as alſo
at the three C, D, M are
ſs than four right angles.
ake therefore [a] a ſolid angle
N contained by the three
ane angles ONP, PNQ,
NQ, which are equal to A,
M, each to each; And by



[a] 23. 11.

op. 26. 11. make at the
int N in the ſtraght line ON a ſolid angle contained by three
ane angles of which one is the angle ONQ equal to M, and
e other two are the angles QNR, ONR which are equal to
e angles C, D, each to each. Thus, at the point N, there
a ſolid angle contained by the four plane angles ONP, PNQ,
NR, ONR which are equal to the angles A, B, C, D, each
each. And that the two ſolid angles at the points F, N,
ch of which is contained by the above named four plane an-
es, are not equal to one another, or that they cannot coin-
de, will be plain by conſidering that the angles GFK, ONQ;
at is, the angles E, M are unequal by the conſtruction; and
erefore the ſtraight lines GF, FK cannot coincide with ON,
Q, nor conſequently can the ſolid angles, which therefore
e unequal.

And becauſe from the four plane angles A, B, C, D, there
n be found innumerable other angles that will ſerve the ſame
rpoſe with the angles E and M; it is plain that innumerable
her ſolid angles may be conſtituted which are each contained
the ſame four plane angles, and all of them unequal to one
other, Q. E. D.

And from this it appears that Clavius and other authors are
ſtaken, who aſſert that thoſe ſolid angles are equal which are
ntained by the ſame number of plane angles that are equal to
e another, each to each. Alſo it is plain that the 26th prop.
book 11 is by no means ſufficiently demonſtrated, becauſe
e equality of two ſolid angles, whereof each is contained by
ee plane angles which are equal to one another, each to each,
only aſſumed, and not demonſtrated.

PROP. I. B. XI.

The words at the end of this, " for a ſtraight line ca
" meet a ſtraight line in more than one point," are left out
an addition by ſome unſkilful hand; for this is to be demon
ted, not aſſumed.

Mr Thomas Simpſon, in his notes at the end of the 2d edi
of his elements of geometry, p. 262. after repeating the wo
of this note, adds, " Now, can it poſſibly ſhew any want
" ſkill in an editor" (he means Euclid or Theon) to refer to
" axiom which Euclid himſelf had laid down, book 1. N° 1
he means Barrow's Euclid, for it is the 10th in the Greek, "
" not to have demonſtrated, what no man can demonſtrate
But all that in this caſe can follow from that axiom is, tha
two ſtraight lines could meet each other in two points, the p
of them betwixt theſe points muſt coincide, and ſo they wo
have a ſegment betwixt theſe points common to both. Now,
it has not been ſhewn in Euclid, that they cannot have a co
mon ſegment, this does not prove that they cannot meet in
points, from which their not having a common ſegment is
duced in the Greek edition: But, on the contrary, becauſe t
cannot have a common ſegment, as is ſhewn in Cor. of
prop. Book 1. of 4to edition, it follows plainly that they can
meet in two points, which the remarker ſays no man can
monſtrate.

Mr Simpſon, in the ſame notes, p. 265. juſtly obſerves, t
in the corollary of prop. 11. book 1. 4to edit. the ſtraight li
AB, BD, BC, are ſuppoſed to be all in the ſame plane, wh
cannot be aſſumed in 1ſt prop. Book 11. This, ſoon after
4to edition was publiſhed, I obſerved and corrected as it is
in this edition: He is miſtaken in thinking the 10th axiom
mentions here to be Euclid's; it is none of Euclid's, but is
10th in Dr Barrow's edition, who had it from Herigon's Cur
vol. 1. and in place of it the corollary of 10. prop. book 1.
added.

PROP. II. B. XI.

This propoſition ſeems to have been changed and vitiated
ſome editor; for all the figures defined in the 1ſt book of
elements, and among them triangles, are, by the hypothe
plane figures; that is, ſuch as are deſcribed in a plane; wh
fore the ſecond part of the enunciation needs no demonſtrati
Beſides, a convex ſuperficies may be terminated by three ſtra

es meeting one another: The thing that ſhould have been monſtrated is, that two, or three ſtraight lines, that meet e another, are in one plane. And as this is not ſufficiently ne, the enunciation and demonſtration are changed into thoſe w put into the text.

PROP. III. B. XI.

In this propoſition the following words near to the end of it left out, viz. "therefore DEB, DFB are not ſtraight lines; in the like manner it may be demonſtrated that there can be no other ſtraight line between the points D, B:" Becauſe m this, that two lines include a ſpace, it only follows that e of them is not a ſtraight line: And the force of the argu- nt lies in this, viz. if the common ſection of the planes be a ſtraight line, then two ſtraight lines could include a ſpace, ich is abſurd; therefore the common ſection is a ſtraight e.

PROP. IV. BOOK XI.

The words "and the triangle AED to the triangle BEC" omitted, becauſe the whole concluſion of the 4th prop. B. has been ſo often repeated in the preceeding books, it was dleſs to repeat it here.

PROP. V. B. XI.

In this, near to the end, ἐπιπέδῳ, ought to be left out in the eek text: And the word "plane" is rightly left out in the ford edition of Commandine's tranſlation.

PROP. VII. B. XI.

This propoſition has been put into this book by ſome un- ful editor, as is evident from this, that ſtraight lines which drawn from one point to another in a plane, are, in the ceeding books, ſuppoſed to be in that plane: And if they e not, ſome demonſtrations in which one ſtraight line is poſed to meet another would not be concluſive, becauſe ſe lines would not meet one another: For inſtance, in prop. B. 1. the ſtraight line GK would not meet EF, if GK were in the plane in which are the parallels AB, CD, and in ich, by hypotheſis, the ſtraight line EF is: Beſides, this propoſition is demonſtrated by the preceeding 3. in which very thing which is propoſed to be demonſtrated in the 7th, wice aſſumed, viz. that the ſtraight line drawn from one nt to another in a plane, is in that plane; and the ſame thing ſſumed in the preceeding 6th prop. in which the ſtraight line

Book XI. which joins the points B, D that are in the plane to which A and CD are at right angles, is ſuppoſed to be in that plane. And the 7th, of which another demonſtration is given, is ke in the book merely to preſerve the number of the propoſitions for it is evident from the 7th and 35th definitions of the book, though it had not been in the elements.

PROP. VIII. B. XI.

In the Greek, and in Commandine's and Dr Gregory's tran lations, near to the end of this propoſition, are the followi words: "But DC is in the plane through BA, AD" inſtead which, in the Oxford edition of Commandine's tranſlation, rightly put "but DC is in the plane through BD, DA:" B "all the editions have the following words, viz. becauſe A "BD are in the plane thro' BD, DA, and DC is in the pla "in which are AB, BD," which are manifeſtly corrupted, have been added to the text; for there was not the leaſt neceſ ty to go ſo far about to ſhew that DC is in the ſame plane which are BD, DA, becauſe it immediately follows from pro 7th preceeding, that BD, DA are in the plane in which are t parallels AB, CD: Therefore, inſtead of theſe words, the ought only to be "becauſe all three are in the plane in whi "are the parallels AB, CD."

PROP. XV. B. XI.

After the words, "and becauſe BA is parallel to GH," t following are added, "for each of them is parallel to DE, a "are not both in the ſame plane with it," as being manifeſ forgotten to be put into the text.

PROP. XVI. B. XI.

In this, near to the end, inſtead of the words, "but ſtraig "lines which meet neither way" ought to be read, "but ſtraig "lines in the ſame plane which produced meet neither way Becauſe, though in citing this definition in prop. 27. Book it was not neceſſary to mention the words, "in the ſame plane all the ſtraight lines in the books preceeding this being in ame plane; yet here it was quite neceſſary.

PROP. XX. B. XI.

In this, near the beginning, are the words, "But if n "let BAC be the greater: But the angle BAC may happ to be equal to one of the other two: Wherefore this place ſho

e read thus, " But if not, let the angle BAC be not leſs than either of the other two, but greater than DAB."

At the end of this propoſition it is ſaid, " in the ſame manner it may be demonſtrated," though there is no need of any emonſtration; becauſe the angle BAC being not leſs than either of the other two, it is evident that BAC together with one f them is greater than the other.

PROP. XXII. B. XI.

And likewiſe in this, near the beginning, it is ſaid, " But if not, let the angles at B, E, H be unequal, and let the angle at B be greater than either of thoſe at E, H:" Which ords manifeſtly ſhew this place to be vitiated, becauſe the ane at B may be equal to one of the other two. They ought erefore to be read thus, " But if not, let the angles at B, E, H be unequal, and let the angle at B be not leſs than either of the other two at E, H: Therefore the ſtraight line AC is not leſs than either of the two DF, GK."

PROP. XXIII. B XI.

The demonſtration of this is made ſomething ſhorter, by not peating in the third caſe the things which were domonſtrated the firſt; and by making uſe of the conſtruction which Camnus has given; but he does not demonſtrate the ſecond and ird caſes: The conſtruction and demonſtration of the third ſe are made a little more ſimple than in the Greek text.

PROP. XXIV. B. XI.

The word " ſimilar" is added to the enunciation of this proſition, becauſe the planes containing the ſolids which are to be emonſtrated to be equal to one another, in the 25th propoſition, ght to be ſimilar and equal, that the equality of the ſolids ay be inferred from prop C. of this book: And, in the Oxrd edition of Commandine's tranſlation, a corollary is added to op. 24. to ſhew that the parallelograms mentioned in this proſition are ſimilar, that the equality of the ſolids in prop. 25. ay be deduced from the 10th def. of book 11.

PROP. XXV. and XXVI. B. XI.

In the 25th prop. ſolid figures which are contained by the me number of ſimilar and equal plane figures, are ſuppoſed be equal to one another. And it ſeems that Theon, or ſome her editor, that he might ſave himſelf the trouble of deonſtrating the ſolid figures mentioned in this propoſition to

Book XI. be equal to one another, has inſerted the 10th def. of this boo to ſerve inſtead of a demonſtration; which was very ignoran done.

Likewiſe in the 26th prop. two ſolid angles are ſuppoſed be equal: If each of them be contained by three plane ang which are equal to one another, each to each. And it is ſtran enough, that none of the commentators on Euclid have, as as I know, perceived that ſomething is wanting in the demo ſtrations of theſe two propoſitions. Clavius, indeed, in a n upon the 11th def. of this book, affirms, that it is evident t thoſe ſolid angles are equal which are contained by the ſa number of plane angles, equal to one another, each to ea becauſe they will coincide, if they be conceived to be pla within one another; but this is ſaid without any proof, nor i always true, except when the ſolid angles are contained by th plane angles only, which are equal to one another, e to each: And in this caſe the propoſition is the ſame with th that two ſpherical triangles that are equilateral to one anoth are alſo equiangular to one another, and can coincide; wh ought not to be granted without a demonſtration. E clid does not aſſume this in the caſe of rectilineal t angles, but demonſtrates in prop. 8. Book 1. that triang which are equilateral to one another are alſo equiangular one another; and from this their total equality appears prop. 4. book 1. And Menelaus, in the 4th prop. of his 1ſt b of ſpherics, explicitly demonſtrates that ſpherical triangles wh are mutually equilateral, are alſo equiangular to one anoth from which it is eaſy to ſhew that they muſt coincide, provid they have their ſides diſpoſed in the ſame order and ſituation

To ſupply theſe defects, it was neceſſary to add the three p poſitions marked A, B, C to this book. For the 25th, 26th, 28th propoſitions of it, and conſequently eight others, viz. 27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſa which depend upon them, have hitherto ſtood upon an inſ foundation; as alſo, the 8th, 12th, Cor. of 17th and 18th 12th book, which depend upon the 9th definition. For it been ſhewn in the notes on def. 10th of this book, that ſ figures which are contained by the ſame number of ſimilar equal plane figures, as alſo ſolid angles that are contained the ſame number of equal plane angles, are not always eq to one another.

It is to be obſerved that Tacquet, in his Euclid, deſines e- al ſolid angles to be ſuch, "as being put within one another do coincide:" But this is an axiom, not a definition; for it true of all magnitudes whatever. He made this uſeleſs defini- n, that by it he might demonſtrate the 36th prop. of this ok, without the help of the 35th of the ſame: Concerning ich demonſtration, ſee the note upon prop. 36.

PROP. XXVIII. B. XI.

In this it ought to have been demonſtrated, not aſſumed, t the diagonals are in one plane. Clavius has ſupplied this fect.

PROP. XXIX. B. XI.

There are three caſes of this propoſition; the firſt is, when two parallelograms oppoſite to the baſe AB have a ſide com- on to both; the ſecond is, when theſe parallelograms are ſe- rated from one another; and the third, when there is a part them common to both; and to this laſt only, the demon- ation that has hitherto been in the elements does agree. The ſt caſe is immediately deduced from the preceeding 28th prop. ich ſeems for this purpoſe to have been premiſed to this 29th, r it is neceſſary to none but to it, and to the 40th of this book, we now have it, to which laſt it would, without doubt, have en premiſed, if Euclid had not made uſe of it in the 29th; t ſome unſkilful editor has taken it away from the elements, d has mutilated Euclid's demonſtration of the other two caſes, ich is now reſtored, and ſerves for both at once.

PROP. XXX. B. XI.

In the demonſtration of this, the oppoſite planes of the ſolid P, in the figure in this edition, that is, of the ſolid CO in ommandine's figure, are not proved to be parallel; which it proper to do for the ſake of learners.

PROP. XXXI. B XI.

There are two caſes of this propoſition; the firſt is, when e inſiſting ſtraight lines are at right angles to the baſes; the her, when they are not: The firſt caſe is divided again into o others, one of which is, when the baſes are equiangular rallelograms; the other, when they are not equiangular:

The

Book XI. The Greek editor makes no mention of the firſt of theſe laſt caſes, but has inſerted the demonſtration of it as a part that of the other: And therefore ſhould have taken no of it in a corollary; but we thought it better to give theſe caſes ſeparately: The demonſtration alſo is made ſometh ſhorter by following the way Euclid has made uſe of in pr 14. book 6. Beſides, in the demonſtration of the caſe in wh the inſiſting ſtraight lines are not at right angles to the ba the editor does not prove that the ſolids deſcribed in the c ſtruction are parallelepipeds, which it is not to be thought t Euclid neglected: Alſo the words, "of which the inſiſti "ſtraight lines are not in the ſame ſtraight lines," have b added by ſome unſkilful hand; for they may be in the ſa ſtraight lines.

PROP. XXXII. B. XI.

The editor has forgot to order the parallelogram FH to applied in the angle FGH equal to the angle LCG, which neceſſary. Clavius has ſupplied this.

Alſo, in the conſtruction, it is required to complete ſolid of which the baſe is FH, and altitude the ſame with t of the ſolid CD; but this does not determine the ſolid to completed, ſince there may be innumerable ſolids upon ſame baſe, and of the ſame altitude: It ought therefore to ſaid "complete the ſolid of which the baſe is FH, and one "its inſiſting ſtraight lines is FD:" The ſame correction m be made in the following propoſition 33.

PROP. D. B. XI.

It is very probable that Euclid gave this propoſition a pl in the Elements, ſince he gave the like propoſition concerni equiangular parallelograms in the 23d B. 6.

PROP. XXXIV. B. XI.

In this the words, ὧν αἱ ἐφεστῶσαι οὐκ εἰσιν ἐπι των αὐτων ἐυθει "of which the inſiſting ſtraight lines are not in the ſa "ſtraight lines," are thrice repeated; but theſe words ou either to be left out, as they are by Clavius, or, in place of the ought to be put, "whether the inſiſting ſtraight lines be, or "not, in the ſame ſtraight lines:" For the other caſe is wi out any reaſon excluded, alſo the words, ὧν τα ὕψη, of wh

he altitudes" are twice put for ων αἱ ἐφεστῶσαι, "of which he inſiſting ſtraight lines;" which is a plain miſtake: For altitude is always at right angles to the baſe.

PROP. XXXV. B. XI.

The angles ABH, DEM are demonſtrated to be right angles a ſhorter way than in the Greek: and in the ſame way ACH, FM may be demonſtrated to be right angles: Alſo the repeon of the ſame demonſtration, which begins with "in the ame manner," is left out, as it was probably added to the t by ſome editor; for the words, "in like manner we may demonſtrate," are not inſerted except when the demonſtration not given, or when it is ſomething different from the other, t be given, as in prop. 26. of this book. Campanus has not s repetition.

We have given another demonſtration of the corollary, bees the one in the original, by help of which the following th prop. may be demonſtrated without the 35.

PROP. XXXVI. B. XI.

Tacquet in his Euclid demonſtrates this propoſition with- t the help of the 35th; but it is plain, that the ſolids mention- in the Greek text in the enunciation of the propoſition equiangular, are ſuch that their ſolid angles are contained three plane angles equal to one another, each to each; as evident from the conſtruction. Now Tacquet does not deonſtrate, but aſſumes theſe ſolid angles to be equal to one other; for he ſuppoſes the ſolids to be already made, and es not give the conſtruction by which they are made: But, the ſecond demonſtration of the preceeding corollary, his monſtration is rendered legitimate likewiſe in the caſe where e ſolids are conſtructed as in the text.

PROP. XXXVII. B. XI.

In this it is aſſumed that the ratios which are triplicate of thoſe tios which are the ſame with one another, are likewiſe the me with one another; and that thoſe ratios are the ſame with e another, of which the triplicate ratios are the ſame with one other; but this ought not to be granted without a demonſtraon, nor did Euclid aſſume the firſt and eaſieſt of theſe two proſitions, but demonſtrated it in the caſe of duplicate ratios, in the d prop. Book 6. On this account, another demonſtration is ven of this propoſition like to that which Euclid gives in prop. . book 6. as Clavius has done.

PROP.

Book XI.

PROP. XXXVIII. B. XI.

When it is required to draw a perpendicular from a poi in one plane which is at right angles to another plane, un this laſt plane, it is done by drawing a perpendicular from th point to the common ſection of the planes; for this perpen dicular will be perpendicular to the plane, by Def. 4. of th book: And it would be fooliſh in this caſe to do it by the 11t prop. of the ſame: But Euclid [a], Apollonius, and other geo meters, when they have occaſion for this problem, direct perpendicular to be drawn from the point to the plane, an conclude that it will fall upon the common ſection of th planes, becauſe this is the very ſame thing as if they had mad uſe of the conſtruction above mentioned, and then conclude that the ſtraight line muſt be perpendicular to the plane; b is expreſſed in fewer words: Some editor, not perceiving thi thought it was neceſſary to add this propoſition, which can nev be of any uſe to the 11th book, and its being near to the end among propoſitions with which it has no connection, is a mark o its having been added to the text.

[a] 17. 12. in other editions.

PROP. XXXIX. B. XI.

In this it is ſuppoſed, that the ſtraight lines which biſect th ſides of the oppoſite planes, are in one plane, which ought t have been demonſtrated; as is now done.

B. XII.

Book XII.

THE learned Mr Moor, profeſſor of Greek in the Univer ſity of Glaſgow, obſerved to me that it plainly appear from Archimedes's epiſtle to Doſitheus, prefixed to his book of the Sphere and Cylinder, which epiſtle he has reſtored fro ancient manuſcripts, that Eudoxus was the author of the chi propoſitions in this 12th book.

PROP. II. B. XII.

At the beginning of this it is ſaid, "if it be not ſo, the ſquar "of BD ſhall be to the ſquare of FH, as the circle ABCD "to ſome ſpace either leſs than the circle EFGH, or greate "than it:" And the like is to be found near to the end of th propoſition, as alſo in prop. 5. 11. 12. 18. of this Book: Con cernin

erning which, it is to be obſerved, that, in the demonſtration of theorems, it is ſufficient, in this and the like caſes, that a thing made uſe of in the reaſoning can poſſibly exiſt, providing this be evident, though it cannot be exhibited or found by a geometrical conſtruction: So, in this place, it is aſſumed, that there may be a fourth proportional to theſe three magnitudes, viz. the ſquares of BD, FH, and the circle ABCD; becauſe it is evident, that there is ſome ſquare equal to the circle ABCD, though it cannot be found geometrically; and to the three rectilineal figures, viz. the ſquares of BD, FH, and the ſquare which is equal to the circle ABCD, there is a fourth ſquare proportional; becauſe to the three ſtraight lines which are their ſides there is a fourth ſtraight line proportional [a], and his fourth ſquare, or a ſpace equal to it, is the ſpace which in this propoſition is denoted by the letter S: And the like is to be underſtood in the other places above cited: And it is probable that this has been ſhewn by Euclid, but left out by ſome editor; for the lemma which ſome unſkilful hand has added to this propoſition explains nothing of it.

[a] 12. 6.

PROP. III. B. XII.

In the Greek text and the tranſlations, it is ſaid, "and becauſe the two ſtraight lines BA, AC which meet one another," &c. here the angles BAC, KHL are demonſtrated to be equal to one another by 10th prop. B. 11. which had been done before: Becauſe the triangle EAG was proved to be ſimilar to the triangle KHL: This repetition is left out, and the triangles BAC, KHL are proved to be ſimilar in a ſhorter way by prop. 21. B. 6.

PROP. IV. B. XII.

A few things in this are more fully explained than in the Greek text.

PROP. V. B. XII.

In this, near to the end, are the words, ὡς ἐμπροσθεν ἐδείχθη, "as was before ſhewn," and the ſame are found again in the end of prop. 18. of this book; but the demonſtration referred to, except it be the uſeleſs lemma annexed to the 2d prop. is no where in theſe elements, and has been perhaps left out by ſome editor who has forgot to cancel thoſe words alſo.

PROP.

PROP. VI. B. XII.

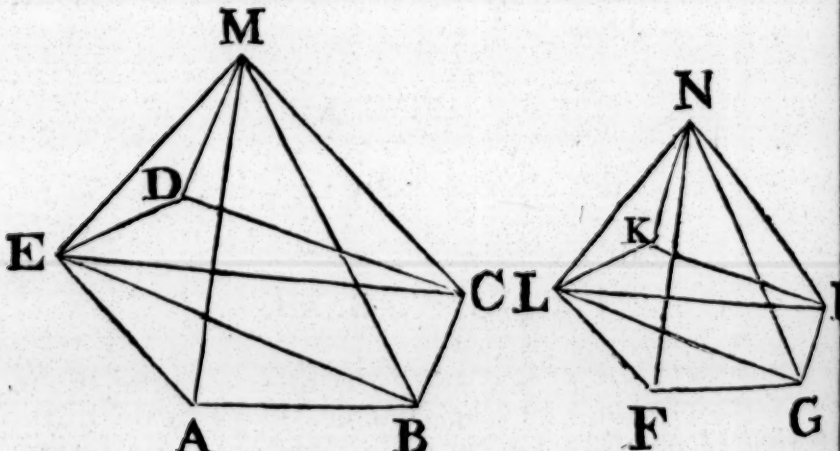
A ſhorter demonſtration is given of this ; and that wh
is in the Greek text may be made ſhorter by a ſtep than it
For the author of it makes uſe of the 22d prop. of B. 5. twic
Whereas once would have ſerved his purpoſe ; becauſe that p
poſition extends to any number of magnitudes which are prop
tionals taken two and two, as well as to three which are p
portional to other three.

COR. PROP. VIII. B. XII.

The demonſtration of this is imperfect, becauſe it is n
ſhewn, that the triangular pyramids into which thoſe upon m
tangular baſes are divided, are ſimilar to one another, as ou
neceſſarily to have been done, and is done in the like caſe
prop. 12. of this book: The full demonſtration of the corolla
is as follows.

Upon the polygonal baſes ABCDE, FGHKL, let there be
milar and ſimilarly ſituated pyramids which have the points
N for their vertices: The pyramid ABCDEM has to the py
mid FGHKLN the triplicate ratio of that which the ſide A
has to the homologous ſide FG.

Let the polygons be divided into the triangles ABE, EB
[a] 20. 6. ECD ; FGL, LGH, LHK, which are ſimilar [a] each to eac
[b] 11. def. And becauſe the pyramids are ſimilar, therefore [b] the trian
11. EAM is ſimilar to the triangle LFN, and the triangle AE
[c] 4. 6. to FGN: Wherefore [c] ME is to EA, as NL to LF ; and as A



to EB, ſo is FL to LG, becauſe the triangles EAB, LFG
ſimilar ; therefore, ex aequali, as ME to EB, ſo is NL to L

like manner it may be ſhewn that EB is to BM, as LG to N; therefore, again, ex aequali, as EM to MB, ſo is LN to G: Wherefore the triangles EMB, LNG having their ſides portionals are [d] equiangular, and ſimilar to one another: erefore the pyramids which have the triangles EAB, LFG their baſes, and the points M, N for their vertices, are ſi- ar [b] to one another, for their ſolid angles are [c] equal, and ſolids themſelves are contained by the ſame number of ſi- ar planes: In the ſame mannner the pyramid EBCM may ſhewn to be ſimilar to the pyramid LGHN, and the pyra- H ECDM to LHKN: And becauſe the pyramids EABM, GN are ſimilar, and have triangular baſes, the pyramid EABM [f] to LFGN the triplicate ratio of that which EB has to the nologous ſide LG. And, in the ſame manner, the pyramid CM has to the pyramid LGHN the triplicate ratio of that ich EB has to LG: Therefore as the pyramid EABM is to pyramid LFGN, ſo is the pyramid EBCM to the pyramid GHN: In like manner, as the pyramid EBCM is to LGHN, is the pyramid ECDM to the pyramid LHKN: And as one the antecedents is to one of the conſequents, ſo are all the ecedents to all the conſequents: Therefore as the pyramid ABM to the pyramid LFGN, ſo is the whole pyramid BCDEM to the whole pyramid FGHKLN: And the pyra- d EABM has to the pyramid LFGN the triplicate ratio of t which AB has to FG; therefore the whole pyramid has he whole pyramid the triplicate ratio of that which AB to the homologous ſide FG. Q. E. D.

Book XII.

[d] 5. 6.

[b] 11. def. 11.

[c] B. 11.

[f] 8. 12.

PROP. XI. and XII. B. XII.

The order of the letters of the alphabet is not obſerved in e two propoſitions, according to Euclid's manner, and is reſtored: By which means the firſt part of prop. 12. may demonſtrated in the ſame words with the firſt part of prop. ; on this account the demonſtration of that firſt part is left , and aſſumed from prop. 11.

PROP. XIII. B. XII.

n this propoſition the common ſection of a plane parallel he baſes of a cylinder, with the cylinder itſelf, is ſuppoſed e a circle, and it was thought proper briefly to demon- te it; from whence it is ſufficiently manifeſt, that this plane des the cylinder into two others: And the ſame thing is un- ood to be ſupplied in prop. 14.

PROP.

PROP. XV. B. XII.

" And complete the cylinders AX, EO," both the enunciation and expoſition of the propoſition repreſent the cylinders as well as the cones as already deſcribed: Wherefore the reading ought rather to be, " and let the cones be ALC, EN " and the cylinders AX, EO."

The firſt caſe in the ſecond part of the demonſtration is wanting; and ſomething alſo in the ſecond caſe of that part, be the repetition of the conſtruction is mentioned; which now added.

PROP. XVII. B. XII.

In the enunciation of this propoſition the Greek words, την μειζονα σφαιραν στερεον πολυεδρον εγγραψαι, μη ψαυον της ελαττ σφαιρας κατα την επιφανειαν, are thus tranſlated by Commandi and others, " in majori ſolidum polyhedrum deſcribere q " minoris ſphaerae ſuperficiem non tangat;" that is, " to " ſcribe in the greater ſphere a ſolid polyhedron which " not meet the ſuperficies of the leſſer ſphere:" Whereby t refer the words κατα την επιφανειαν to theſe next to them ελασσονος σφαιρας: But they ought by no means to be thus tr lated, for the ſolid polyhedron doth not only meet the ſu ficies of the leſſer ſphere, but pervades the whole of that ſphe Therefore the foreſaid words are to be referred to το στ πολυεδρον, and ought thus to be tranſlated, viz. to deſcribe the greater ſphere a ſolid polyhedron whoſe ſuperficies ſhall meet the leſſer ſphere; as the meaning of the propoſition ceſſarily requires.

The demonſtration of the propoſition is ſpoiled and m lated: For ſome eaſy things are very explicitly demonſtra while others not ſo obvious are not ſufficiently explained; example, when it is affirmed, that the ſquare of KB is gre than the double of the ſquare of BZ, in the firſt demonſ tion; and that the angle BZK is obtuſe, in the ſecond: B which ought to have been demonſtrated: Beſides, in the firſt monſtration, it is ſaid, " draw KΩ from the point K perp " dicular to BD;" whereas it ought to have been ſaid, " " KV," and it ſhould have been demonſtrated that KV perpendicular to BD: For it is evident from the figure in H vagius's and Gregory's editions, and from the words of

demonſtra

monſtration, that the Greek Editor did not perceive that e perpendicular drawn from the point K to the ſtraight line D muſt neceſſarily fall upon the point V, for in the figure it made to fall upon the point Ω a different point from V, hich is likewiſe ſuppoſed in the demonſtration. Commane ſeems to have been aware of this; for in this figure he arks one and the ſame point with the two letters V, Ω; and fore Commandine, the learned John Dee in the commenry he annexes to this propoſition in Henry Billinſley's tranion of the Elements printed at London, Ann. 1570, exeſsly takes notice of this error, and gives a demonſtration ited to the conſtruction in the Greek text, by which he ſhews at the perpendicular drawn from the point K to BD, muſt neſſarily fall upon the point V.

Likewiſe it is not demonſtrated that the quadrilateral figures OPT, TPRY, and the triangle YRX do not meet the leſſer here, as was neceſſary to have been done: Only Clavius, as r as I know, has obſerved this, and demonſtrated it by a lem, which is now premiſed to this propoſition, ſomething alted and more briefly demonſtrated.

In the corollary of this propoſition it is ſuppoſed that a lid polyhedron is deſcribed in the other ſphere ſimilar to that hich is deſcribed in the ſphere BCDE; but, as the conſtruction which this may be done is not given, it was thought proper give it, and to demonſtrate that the pyramids in it are ſimilar thoſe of the ſame order in the ſolid polyhedron deſcribed in the here BCDE.

From the preceeding notes, it is ſufficiently evident how uch the Elements of Euclid, who was a moſt accurate geeter, have been vitiated and mutilated by ignorant editors. he opinion which the greateſt part of learned men have entained concerning the preſent Greek edition, viz. that it is ery little or nothing different from the genuine work of Euid, has without doubt deceived them, and made them leſs ttentive and accurate in examining that edition; whereby ſeeral errors, ſome of them groſs enough, have eſcaped their otice from the age in which Theon lived to this time. Upon hich account there is ſome ground to hope that the pains we ave taken in correcting thoſe errors, and freeing the elements as far we could from blemiſhes, will not be unac-

Book XII. ceptable to good judges who can diſcern when demonſtrat
are legitimate, and when they are not.

The objections which, ſince the firſt edition, have
made againſt ſome things in the notes, eſpecially againſt
doctrine of proportionals, have either been fully anſwere
Dr Barrow's Lect. Mathemat. and in theſe notes; or are
except one which has been taken notice of in the note
Prop. 1. Book 11. as ſhew that the perſon who made t
has not ſufficiently conſidered the things againſt which
are brought; ſo that it is not neceſſary to make any fur
anſwer to theſe objections and others like them againſt Euc
definition of proportionals; of which definition Dr Ba
juſtly ſays in page 297. of the above named Book, that "
" ſi machinis impulſa validioribus aeternum perſiſter
" cuſſa."

FINIS.

EUCLID's DATA.

IN THIS EDITION

SEVERAL ERRORS ARE CORRECTED,

AND

SOME PROPOSITIONS ADDED.

BY

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PREFACE.

EUCLID'S DATA is the firſt in order of the books written by the ancient geometers to facilitate and promote the method of reſolution or analyſis. In the general, a thing is ſaid to be given which is either actually exhibited, or can be found out, that is, which is either known by hypotheſis, or that can be demonſtrated to be known; and the propoſitions in the book of Euclid's Data ſhew what things can be found out or known from thoſe that by hypotheſis are already known; ſo that in the analyſis or inveſtigation of a problem, from the things that are laid down to be known or given, by the help of theſe propoſitions other things are demonſtrated to be given, and from theſe, other things are again ſhewn to be given, and ſo on, until that which was propoſed to be found out in the problem is demonſtrated to be given, and when this is done, the problem is ſolved, and its compoſition is made and derived from the compoſitions of the Data which were made uſe of in the analyſis. And thus the Data of Euclid are of the moſt general and neceſſary uſe in the ſolution of problems of every kind.

Euclid is reckoned to be the author of the Book of the Data, both by the ancient and modern geometers; and there ſeems to be no doubt of his having written a book on this ſubject, but which, in the courſe of ſo many ages, has been much vitiated by unſkilful editors in ſeveral places, both in the order of the propoſitions, and in the definitions and demonſtrations themſelves. To correct the errors which are now found in it, and bring it nearer to the accuracy with which it was, no doubt, at firſt written by Euclid, is the deſign of this edition, that ſo it may be rendered more uſeful to geome-

ters, at leaſt to beginners who deſire to learn the inveſtigatory method of the ancients. And for their ſakes, the compoſitions of moſt of the Data are ſubjoined to their demonſtrations, that the compoſitions of problems ſolved by help of the data may be the more eaſily made.

Marinus the philoſopher's preface, which, in the Greek edition is prefixed to the Data, is here left out, as being of no uſe to underſtand them. At the end of it, he ſays, that Euclid has not uſed the ſynthetical, but the analytical method in delivering them; in which he is quite miſtaken; for in the analyſis of a theorem the thing to be demonſtrated is aſſumed in the analyſis; but in the demonſtrations of the data, the thing to be demonſtrated, which is, that ſomething or other is given, is never once aſſumed in the demonſtration, from which it is manifeſt, that every one of them is demonſtrated ſynthetically; though indeed, if a propoſition of the Data be turned into a problem, for example the 84th or 85th in the former editions, which here are the 85th and 86th, the demonſtration of the propoſition becomes the analyſis of the problem.

Wherein this edition differs from the Greek, and reaſons of the alterations from it, will be ſhewn in the notes at the end of the Data.

EUCLID'S

EUCLID'S DATA.

DEFINITIONS.

I.

SPACES, lines, and angles, are ſaid to be given in magnitude, when equals to them can be found.

II

A ratio is ſaid to be given, when a ratio of a given magnitude to a given magnitude which is the ſame ratio with it can be found.

III.

Rectilineal figures are ſaid to be given in ſpecies, which have each of their angles given, and the ratios of their ſides given.

IV.

Points, lines, and ſpaces, are ſaid to be given in poſition, which have always the ſame ſituation, and which are either actually exhibited, or can be found.

A.

An angle is ſaid to be given in poſition, which is contained by ſtraight lines given in poſition.

V.

A circle is ſaid to be given in magnitude, when a ſtraight line from its center to the circumference is given in magnitude.

VI.

A circle is ſaid to be given in poſition and magnitude, the center of which is given in poſition, and a ſtraight line from it to the circumference is given in magnitude.

VII.

Segments of circles are ſaid to be given in magnitude, when the angles in them, and their baſes are given in magnitude.

VIII.

Segments of circles are ſaid to be given in poſition and magnitude, when the angles in them are given in magnitude, and their baſes are given both in poſition and magnitude.

IX.

A magnitude is ſaid to be greater than another by a given magnitude, when this given magnitude being taken from it, the remainder is equal to the other magnitude.

X.

A magnitude is said to be less than another by a given magnitude, when this given magnitude being added to it, the whole is equal to the other magnitude.

* 1. See N.

PROPOSITION I.

THE ratios of given magnitudes to one another is given.

Let A, B be two given magnitudes, the ratio of A to B given.

[a] 1. def. dat. Because A is a given magnitude, there may [a] be found one equal to it; let this be C: And because B is given, one equal to it may be found; let it be D; and since A is equal to C, and B to D; therefore [b] A is to B, as C to D; and consequently the ratio of A to B is given, because the ratio of the given magnitudes C, D which is the same with it has been found.

[b] 7. 5.

2. See N.

PROP. II.

IF a given magnitude has a given ratio to another magnitude, "and if unto the two magnitudes by "which the given ratio is exhibited, and the given "magnitude, a fourth proportional can be found, the other magnitude is given.

Let the given magnitude A have a given ratio to the magnitude B; if a fourth proportional can be found to the three magnitudes above named, B is given in magnitude.

[a] 1. def. Because A is given, a magnitude may be found equal to it [a]; let this be C: And because the ratio of A to B is given, a ratio which is the same with it may be found, let this be the ratio of the given magnitude E to the given magnitude F: Unto the magnitudes E, F, C find a fourth proportional D, which, by the hypothesis, can be done. Wherefore, because A is to B, as E to F; and as E to F, so is C to D: A is [b] to B, as C to

[b] 11. 5.

* The figures in the margin shew the number of the propositions in the other editions.

But A is equal to C; therefore [c] B is equal to D. The magnitude B is therefore given [a], becauſe a magnitude D equal to it has been found. [c] 14. 5. [a] 1. def.

The limitation within the inverted commas is not in the Greek text, but is now neceſſarily added; and the ſame muſt be underſtood in all the propoſitions of the book which depend upon this ſecond propoſition, where it is not expreſsly mentioned. See the note upon it.

PROP. III. 3.

IF any given magnitudes be added together, their ſum ſhall be given.

Let any given magnitudes AB, BC be added together, their ſum AC is given.

Becauſe AB is given, a magnitude equal to it may be found [a]; let this be DE: And becauſe BC is given, one equal to it may be found; let this be EF: Wherefore, becauſe AB is equal to DE, and BC equal to EF; the whole AC is equal to the whole DF: AC is therefore given, becauſe DF has been found which is equal to it. [a] 1. def.

A B C
D E F

PROP. IV. 4.

IF a given magnitude be taken from a given magnitude; the remaining magnitude ſhall be given.

From the given magnitude AB, let the given magnitude AC be taken; the remaining magnitude CB is given.

Becauſe AB is given, a magnitude equal to it may [a] be found; let this be DE: And becauſe AC is given, one equal to it may be found; let this be DF: Wherefore, becauſe AB is equal to DE, and AC to DF; the remainder CB is equal to the remainder FE. CB is therefore given [a], becauſe FE which is equal to it has been found. [a] 1. def.

A C B
D F E

PROP

12.

PROP. V.

See N. IF of three magnitudes, the first together with the second be given, and also the second together with the third; either the first is equal to the third, or one of them is greater than the other by a given magnitude.

Let AB, BC, CD be three magnitudes, of which AB together with BC, that is AC, is given; and also BC together with CD, that is BD, is given. Either AB is equal to CD, or one of them is greater than the other by a given magnitude.

Because AC, BD are each of them given, they are either equal to one another, or not equal. First, let them be equal, and because AC is equal to BD, take away the common part BC; therefore the remainder AB is equal to the remainder CD.

A B C D

But if they be unequal, let AC be greater than BD, and make CE equal to BD. Therefore CE is given, because BD is given. And the whole AC is given; therefore [a] AE the remainder is given. And because EC is equal to BD, by taking BC from both, the remainder EB is equal to the remainder CD. And AE is given; wherefore AB exceeds EB, that is CD, by the given magnitude AE.

[a] 4. dat.

A E B C D

PROP. VI.

5.

See N. IF a magnitude has a given ratio to a part of it, it shall also have a given ratio to the remaining part of it.

Let the magnitude AB have a given ratio to AC a part of it; it has also a given ratio to the remainder BC.

Because the ratio of AB to AC is given, a ratio may be found [a], which is the same to it: Let this be the ratio of DE a given magnitude to the given magnitude DF. And because DE, DF are given, the remainder FE is [b] given. And because AB is to AC, as DE to DF, by conversion [c] AB is to BC, as DE to EF. Therefore the ratio of AB to BC is given, because the ratio of the given magnitudes DE, EF, which is the same with it, has been found.

[a] 2. def.

[b] 4. dat.

[c] E. 5.

A C B

D F E

Cor.

COR. From this it follows, that the parts AC, CB have a gi- ratio to one another: Because as AB to BC, ſo is DE to by diviſion [d], AC is to CB, as DF to FE; and DF, FE given; therefore [a] the ratio of AC to CB is given. [d] 17. 5. [a] 2. def.

PROP. VII. 6.

F two magnitudes which have a given ratio to one another, be added together; the whole magnitude ſhall have to each of them a given ratio. See N.

Let the magnitudes AB, BC which have a given ratio to one ther, be added together; the whole AC has to each of the gnitudes AB, BC a given ratio.

Becauſe the ratio of AB to BC is given, a ratio may be nd [a] which is the ſame with it; let this be the ratio of [a] 2. def.
given magnitudes DE, EF: And
auſe DE, DF are given, the whole
is given [b]: And becauſe as AB to
, ſo is DE to EF; by compoſition
C is to CB, as DF to FE; and by
verſion [d], AC is to AB, as DF to DE: Wherefore becauſe [d] E. 5.
C is to each of the magnitudes AB, BC, as DF to each of others DE, EF; the ratio of AC to each of the magnitudes , BC is given [a].

A B C

D E F

[b] 3. Dat.

[c] 18. 5.

PROP. VIII. 7.

F a given magnitude be divided into two parts which have a given ratio to one another, and if a fourth proportional can be found to the ſum of the two magnitudes by which the given ratio is exhibited, one of them, and the given magnitude; each of the parts given. See N.

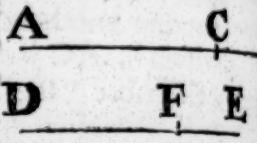
Let the given magnitude AB be divided into the parts AC, which have a given ratio to one another; if a fourth portional can be found to the above med magnitudes; AC and CB are ch of them given.

Becauſe the ratio of AC to CB is
ven, the ratio of AB to BC is given [a]; therefore a ratio [a] 7. Dat.
which

A C B

D F E

[b] 2. def. which is the ſame with it can be found [b], let this be the r
of the given magnitudes DE, EF:
And becauſe the given magnitude AB
has to BC the given ratio of DE to
EF, if unto DE, EF, AB a fourth
proportional can be found, this which
[c] 2. Dat is BC is given [c]; and becauſe AB is given, the other part A
[d] 4. Dat. given [d].



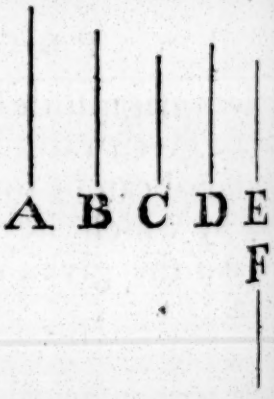
In the ſame manner, and with the like limitation, if
difference AC of two magnitudes AB, BC which have a gi
ratio be given; each of the magnitudes AB, BC is given.

8. PROP. IX.

MAGNITUDES which have given ratios to
ſame magnitude, have alſo a given ratio
one another.

Let A, C have each of them a given ratio to B; A has a
ven ratio to C.

Becauſe the ratio of A to B is given, a ratio which is
ſame to it may be found [a]; let this be the ratio of the gi
magnitudes D, E: And becauſe the ratio of B to C is given
[a] 2. Def. ratio which is the ſame with it may be found [a]; let this be
ratio of the given magnitudes F, G:
To F, G, E find a fourth propor-
tional H, if it can be done; and
becauſe as A is to B, ſo is D to E;
and as B to C, ſo is (F to G, and
ſo is) E to H; ex aequali, as A to
C, ſo is D to H: Therefore the ratio
of A to C is given [a], becauſe the
ratio of the given magnitudes D and
H, which is the ſame with it, has
been found: But if a fourth propor-
tional to F, G, E cannot be found,
then it can only be ſaid that the ratio of A to C is compoun
of the ratios of A to B, and B to C, that is, of the given ra
of D to E, and F to G.



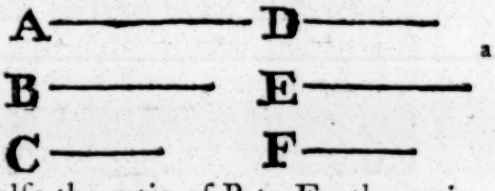
PR

PROP. X. 9.

f two or more magnitudes have given ratios to one another, and if they have given ratios, though ey be not the ſame, to ſome other magnitudes; eſe other magnitudes ſhall alſo have given ratios one another.

Let two or more magnitudes A, B, C have given ratios to e another; and let them have given ratios, though they be not e ſame, to ſome other magnitudes D, E, F: The magnitudes , E, F have given ratios to one another.

Becauſe the ratio of A to B is given, and likewiſe the ratio A to D; therefore the ra- of D to B is given [a]; but e ratio of B to E is given, refore [a] the ratio of D to is given: And becauſe the io of B to C is given, and alſo the ratio of B to E; the ratio E to C is given [a]: And the ratio of C to F is given; where- e the ratio of E to F is given: D, E, F have therefore given ra- s to one another.



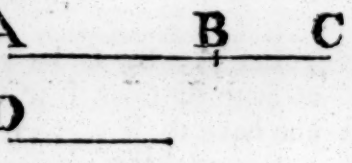
[a] 9. Dat.

PROP. XI. 22.

F two magnitudes have each of them a given ratio to another magnitude; both of them together all have a given ratio to that other.

Let the magnitudes AB, BC have a given ratio to the magni- de D; AC has a given ratio to the ſame D.

Becauſe AB, BC have each of em a given ratio to D, the ratio AB to BC is given: And by mpoſition, the ratio of AC to CB given [b]: But the ratio of BC to s given; therefore [a] the ratio of AC to D is given.



[a] 9. Dat.

[b] 7. Dat.

PROP.

23.

PROP. XII.

See N. IF the whole have to the whole a given ratio, the parts have to the parts given, but not ſame, ratios: Every one of them, whole or part, have to every one a given ratio.

Let the whole AB have a given ratio to the whole CD, the parts AE, EB have given, but not the ſame, ratios t parts CF, FD: Every one ſhall have to every one, who part, a given ratio.

Becauſe the ratio of AE to CF is given, as AE to C make AB to CG; the ratio therefore of AB to CG is gi wherefore the ratio of the remainder EB to the rema
[a] 19. 5. FG is given, becauſe it is the ſame [a] with the ratio of A CG; And the ratio of EB to FD is given, wherefore the ratio of FD
[b] 9. Dat. to FG is given [b]; and by converſion, the ratio of FD to DG is
[c] 6. Dat. given [c]: And becauſe AB has to each of the magnitudes CD, CG a given ratio, the ratio o to CG is given [b]; and therefore [c] the ratio of CD to D given: But the ratio of GD to DF is given, wherefore
[d] Cor. 6. Dat. ratio of CD to DF is given, and conſequently [d] the ratio o to FD is given; but the ratio of CF to AE is given, as alſ
[e] 10. Dat. ratio of FD to EB; wherefore [e] the ratio of AE to EB is gi
[f] 7. Dat. as alſo the ratio of AB to each of them [f]: The ratio ther of every one to every one is given.

A E
C F G

24.

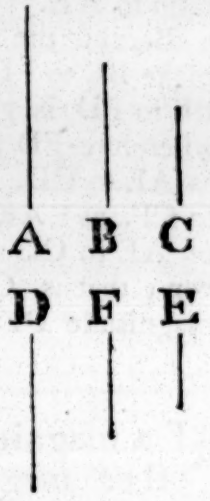
PROP. XIII.

See N. IF the firſt of three proportional ſtraight lines h given ratio to the third, the firſt ſhall alſo ha given ratio to the ſecond.

Let A, B, C be three proportional ſtraight lines, th as A to B, ſo is B to C; if A has to C a given ratio, A alſo have to B a given ratio.

Becauſe the ratio of A to C is given, a ratio which i
[a] 2. Def. ſame with it may be found [a]; let this be the ratio of th
[b] 13. 6. ven ſtraight lines D, E; and between D and E find a [b] propor

roportional F; therefore the rectangle contained by D and is equal to the ſquare of F, and the rectangle D, E is given becauſe its ſides D, E are iven; wherefore the ſquare of F, and the traight line F is given: And becauſe as A is o C, ſo is D to E; but as A to C, ſo is [c] the quare of A to the ſquare of B; and as D to , ſo is [c] the ſquare of D to the ſquare of F; herefore the ſquare [d] of A is to the ſquare of , as the ſquare of D to the ſquare of F: s therefore [e] the ſtraight line A to the traight line B, ſo is the ſtraight line D to the traight line F: Therefore the ratio of A to B given [a], becauſe the ratio of the given traight lines D, F which is the ſame with it as been found.



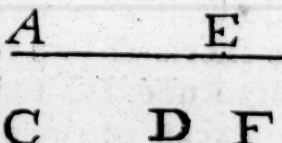
[c] 2. Cor. 20. 6.
[d] 11. 5.
[e] 22. 6.
[a] 2. def.

PROP. XIV. A.

IF a magnitude together with a given magnitude has a given ratio to another magnitude; the exceſs of this other magnitude above a given magnitude has a given ratio to the firſt magnitude: And if the exceſs of a magnitude above a given magnitude has a given ratio to another magnitude; this other magnitude together with a given magnitude has a given ratio to the firſt magnitude. See N.

Let the magnitude AB together with the given magnitude BE, that is AE, have a given ratio to the magnitude CD; the exceſs of CD above a given magnitude has a given ratio to AB.

Becauſe the ratio of AE to CD is given, as AE to CD, ſo make BE to FD; therefore the ratio of BE to FD is given, and BE is given, wherefore FD is given [a]: And becauſe as AE to CD, ſo is BE to FD, the remainder AB is [b] to the remainder CF, as AE to CD: But the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is, CF the exceſs of CD above the given magnitude FD has a given ratio to AB.

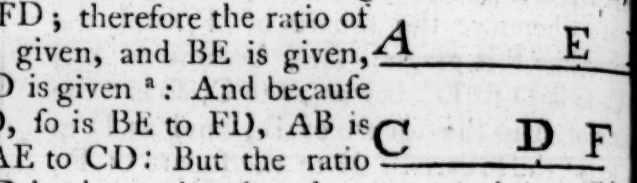


[a] 2. dat.
[b] 19. 5.

Next, Let the exceſs of the magnitude AB above the given magnitude BE, that is, let AE have a given ratio to the magnitude

nitude CD; CD together with a given magnitude has a giv ratio to AB.

Becauſe the ratio of AE to CD is given, as AE to CD, make BE to FD; therefore the ratio of BE to FD is given, and BE is given, wherefore FD is given [a]: And becauſe as AE to CD, ſo is BE to FD, AB is to CF, as [c] AE to CD: But the ratio of AE to CD is given, therefore the ratio of AB to CF is ven; that is, CF which is equal to CD together with the gi magnitude DF has a given ratio to AB.



[a] 2. dat.

[c] 12. 5.

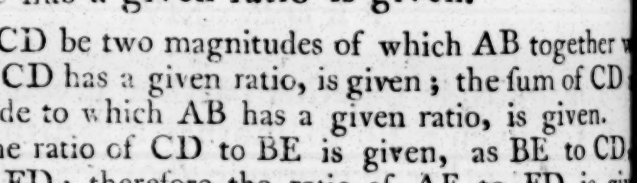
B. See N.

PROP. XV.

IF a magnitude together with that to which an ther magnitude has a given ratio, be given the ſum of this other, and that to which the f magnitude has a given ratio is given.

Let AB, CD be two magnitudes of which AB together w BE to which CD has a given ratio, is given; the ſum of CD that magnitude to which AB has a given ratio, is given.

Becauſe the ratio of CD to BE is given, as BE to CD make AE to FD; therefore the ratio of AE to FD is giv and AE is given, wherefore [a] FD is given: And becauſe as BE to CD, ſo is AE to FD; AB is [b] to FC, as BE to CD: And the ratio of BE to CD is given, wherefore the ratio of AB to FC is given: And FD is given, that is, together with FC to which AB has a given ratio is given.



[a] 2. dat.

[b] Cor. 19. 5.

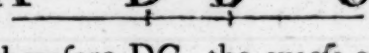
10. See N.

PROP. XVI.

IF the exceſs of a magnitude above a given ma nitude, has a given ratio to another magnitu the exceſs of both together above the given mag tude ſhall have to that other a given ratio: An the exceſs of two magnitudes together above a giv magnitude, has to one of them a given ratio; eit the exceſs of the other above a given magnitude to that one a given ratio; or the ſum of the oth and the magnitude to which that one has a given tio, is given.

Let the exceſs of the magnitude AB above a given magnitude, have a given ratio to the magnitude BC; the exceſs of AC, both of them together, above the given magnitude, has a given ratio to BC.

Let AD be the given magnitude, the exceſs of AB above which, viz. DB, has a given ratio to BC: And becauſe DB has a given ratio to BC, the ratio of DC to



BC is given [a], and AD is given; therefore DC, the exceſs of AC above the given magnitude AD, has a given ratio to BC. [a] 7. dat.

Next, let the exceſs of two magnitudes AB, BC together, above a given magnitude, have to one of them BC a given ratio; either the exceſs of the other of them

AB above the given magnitude ſhall have to BC a given ratio; or AB, together with the magnitude to which BC has a given ratio, is given.

Let AD be the given magnitude, and firſt let it be leſs than AB; and becauſe DC the exceſs of AC above AD has a given ratio to BC, DB has [b] a given ratio to BC; that is, DB, the exceſs of AB above the given magnitude AD, has a given ratio to BC. [b] Cor. 6. dat.

But let the given magnitude be greater than AB, and make AE equal to it; and becauſe EC, the exceſs of AC above AE, has to BC a given ratio, BC has [c] a given ratio to BE; and becauſe AE is given, AB, together with BE to which BC has a given ratio, is given. [c] 6. dat.

PROP. XVII. 11.

If the exceſs of a magnitude above a given magnitude has a given ratio to another magnitude; the exceſs of the ſame firſt magnitude above a given magnitude, ſhall have a given ratio to both the magnitudes together: And if the exceſs of either of two magnitudes above a given magnitude has a given ratio to both magnitudes together; the exceſs of the ſame above a given magnitude ſhall have a given ratio to the other. See N.

Let the exceſs of the magnitude AB above a given magnitude have a given ratio to the magnitude BC; the exceſs of AB above a given magnitude has a given ratio to AC.

Let AD be the given magnitude; and becauſe DB, the ex-
ceſs of AB above AD, has a given ratio to BC; the ratio
[a] 7. dat. DC to DB is given [a]: Make the ratio of AD to DE the ſa
with this ratio; therefore the ratio
of AD to DE is given: And AD A E D B
[b] 2. dat. is given, wherefore [b] DE, and the remainder AE are given: A
[c] 12. 5. becauſe as DC to DB, ſo is AD to DE, AC is [c] to EB, as D
to DB; and the ratio of DC to DB is given; wherefore
ratio of AC to EB is given: And becauſe the ratio of EB to A
is given, and that AE is given, therefore EB the exceſs of
above the given magnitude AE, has a given ratio to AC.

Next, let the exceſs of AB above a given magnitude hav
given ratio to AB and BC together, that is, to AC; the exc
of AB above a given magnitude has a given ratio to BC.

Let AE be the given magnitude; and becauſe EB the exc
of AB above AE has to AC a given ratio, as AC to EB,
make AD to DE; therefore the ratio of AD to DE is give
[d] 6. dat. as alſo [d] the ratio of AD to AE: And AE is given, whe
fore [b] AD is given: And becauſe, as the whole AC, to
[e] 19. 5. whole EB, ſo is AD to DE; the remainder DC is [e] to
remainder DB, as AC to EB; and the ratio of AC to EB
[f] Cor. 6. dat. given; wherefore the ratio of DC to DB is given, as alſo [f]
ratio of DB to BC: And AD is given, therefore DB, the
ceſs of AB above a given magnitude AD, has a given ra
to BC.

14. PROP. XVIII.

IF to each of two magnitudes, which have a giv
ratio to one another, a given magnitude be ad
ed; the whole ſhall either have a given ratio to o
another, or the exceſs of one of them above a giv
magnitude ſhall have a given ratio to the other.

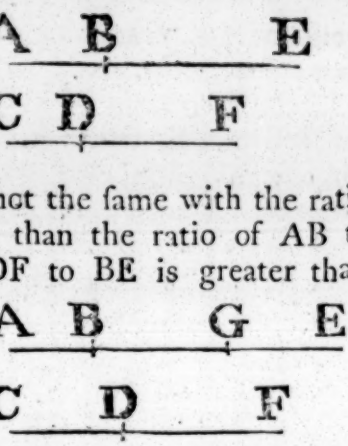
Let the two magnitudes AB, CD have a given ratio to
another, and to AB let the given magnitude BE be added,
the given magnitude DF to CD: The wholes AE, CF eit
have a given ratio to one another, or the exceſs of one of th
[a] 1. dat. above a given magnitude has a given ratio to the other [a].

Becauſe BE, DF are each of them given, their ratio is giv

nd if this ratio be the ſame with
he ratio of AB to CD, the ratio of
AE to CF, which is the ſame [b] with
he given ratio of AB to CD, ſhall
e given.

[b] 12. 5.

But if the ratio of BE to DF, be not the ſame with the ratio f AB to CD; either it is greater than the ratio of AB to CD, or, by inverſion, the ratio of DF to BE is greater than he ratio of CD to AB: Firſt, let he ratio of BE to DF be greater han the ratio of AB to CD; and as AB to CD, ſo make BG to DF; herefore the ratio of BG to DF is iven; and DF is given, therefore [c] BG is given: And becauſe E has a greater ratio to DF than (AB to CD, that is, than) G to DF, BE is greater [d] than BG: And becauſe as AB to D, ſo is BG to DF; therefore AG is [b] to CF, as AB to CD: ut the ratio of AB to CD is given, wherefore the ratio of AG CF is given; and becauſe BE, BG are each of them given, E is given: Therefore AG, the exceſs of AE above a given agnitude GE, has a given ratio to CF. The other caſe is deonſtrated in the ſame manner.



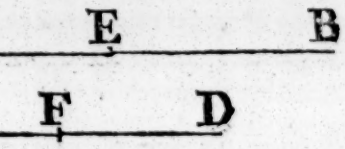
[c] 2. dat.

[d] 10. 5.

PROP. XIX. 15.

F from each of two magnitudes, which have a given ratio to one another, a given magnitude be ken; the remainder ſhall either have a given ratio one another, or the exceſs of one of them above given magnitude, ſhall have a given ratio to the ther.

Let the magnitudes AB, CD have a given ratio to one anher, and from AB let the given magnitude AE be taken, d from CD, the given magnitude CF: The remainders EB, D ſhall either have a given ratio to one another, or the exs of one of them above a gi- magnitude ſhall have a given o to the other.

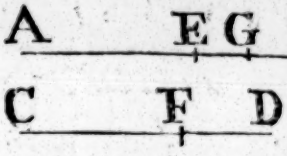


Becauſe AE, CF are each of em given, their ratio is given [a]; d if this ratio be the ſame with the ratio of AB to CD, the

[a] 1. dat.

ratio of the remainder EB to the remainder FD, which is
[b] 19. 5. ſame [b] with the given ratio of AB to CD, ſhall be given.

But if the ratio of AB to CD be not the ſame with the ra
of AE to CF, either it is greater than the ratio of AE to C
or, by inverſion, the ratio of CD to AB is greater than
ratio of CF to AE: Firſt, let the ratio of AB to CD be gre
than the ratio of AE to CF, and as AB to CD, ſo make A
to CF; therefore the ratio of AG
to CF is given, and CF is given,
[c] 2. dat. wherefore [c] AG is given: And
becauſe the ratio of AB to CD,
that is, the ratio of AG to CF,
[d] 10. 5. is greater than the ratio of AE to CF; AG is greater [d] t
AE: And AG, AE, are given, therefore the remainder
is given; and as AB to CD, ſo is AG to CF, and ſo is [b] the
mainder GB to the remainder FD; and the ratio of AB to
is given: Wherefore the ratio of GB to FD is given; th
fore GB, the exceſs of EB above a given magnitude EG, h
given ratio to FD. In the ſame manner the other caſe is
monſtrated.

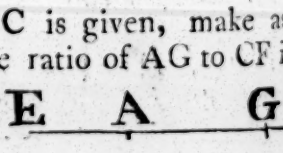


16. PROP. XX.

IF to one of two magnitudes which have a gi
ratio to one another, a given magnitude be a
ed, and from the other a given magnitude be tak
the exceſs of the ſum above a given magnitude ſh
have a given ratio to the remainder.

Let the two magnitudes AB, CD have a given ratio to
another, and to AB let the given magnitude EA be added,
from CD let the given magnitude CF be taken; the excel
the ſum EB above a given magnitude has a given ratio to
remainder FD.

Becauſe the ratio of AB to DC is given, make as A
CD, ſo AG to CF: Therefore the ratio of AG to CF is gi
[a] 2. dat. and CF is given, wherefore [a] AG
is given; and EA is given, there-
fore the whole EG is given: And
becauſe as AB to CD, ſo is AG
[b] 19. 5. to CF, and ſo is [b] the remainder
GB to the remainder FD; the ratio of GB to FD is gi
And EG is given, therefore GB, the exceſs of the ſum E



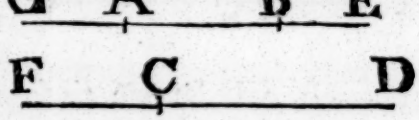
re a given magnitude EG, has a given ratio to the remain-
r FD.

PROP. XXI. C.

F two magnitudes have a given ratio to one ano- See N.
ther, if a given magnitude be added to one of
em, and the other be taken from a given magni-
de; the ſum, together with the magnitude to which
e remainder has a given ratio, is given; And the
mainder, together with the magnitude to which
e ſum has a given ratio, is given.

Let the two magnitudes AB, CD have a given ratio to one
ther; and to AB let the given magnitude BE be added,
d let CD be taken from the given magnitude FD: The ſum
E is given together with the magnitude to which the re-
inder FC has a given ratio.

Becauſe the ratio of AB to CD is given, make as AB to
, ſo GB to FD: Therefore the ratio of GB to FD is given
d FD is given, wherefore GB
given [a]; and BE is given, the
ole GE is therefore given: And
auſe as AB to CD, ſo is GB
FD, and ſo is [b] GA to FC; the
io of GA to FC is given: And AE together with GA is gi-
n, becauſe GE is given; therefore the ſum AE together with
A, to which the remainder FC has a given ratio, is given.
e ſecond part is manifeſt from prop. 15.



[a] 2. dat.

[b] 19. 5.

PROP. XXII. D.

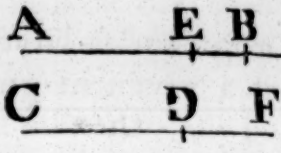
F two magnitudes have a given ratio to one ano- See N.
ther, if from one of them a given magnitude be
ken, and the other be taken from a given magni-
de; the ſum of each of the remainders, and the
agnitude to which the other remainder has a given
tio, is given.

Let the two magnitudes AB, CD have a given ratio to one
other, and from AB let the given magnitude AE be taken,

and let CD be taken from the given magnitude CF; the su of the remainder EB, and the magnitude to which the oth remainder DF has a given ratio, is given.

Becauſe the ratio of AB to CD is given, make as AB CD, ſo AG to CF: The ratio of AG to CF is therefore give

a 2. dat. and CF is given, wherefore a AG is given; and AE is given, and therefore the remainder EG is given: And becauſe as AB to CD,



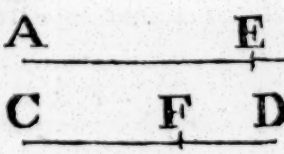
b 19. 5. ſo is AG to CF: And ſo is b the remainder BG to the remainder DF; the ratio of BG to DF given: And EB together with BG is given, becauſe EG is given: Therefore the ſum of the remainder EB, and BG to whi DF the other remainder has a given ratio is given. The ſecon part is plain from this and prop. 15.

20. PROP. XXIII.

See N. IF, from two given magnitudes there be taken ma nitudes which have a given ratio to one anothe the remainders ſhall either have a given ratio to on another, or the exceſs of one of them above a give magnitude ſhall have a given ratio to the other.

Let AB, CD be two given magnitudes, and from them the magnitudes AE, CF, which have a given ratio to one an ther, be taken; the remainders EB, FD either have a given ra to one another, or the exceſs of one of them above a given ma nitude has a given ratio to the other.

Becauſe AB, CD are each of them given, the ratio of AB to CD is given: And if this ratio be the ſame with the ratio of AE to CF, then the remainder EB

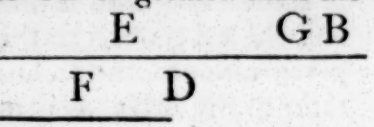


a 19. 5. has a the ſame given ratio to the remainder FD.

But if the ratio of AB to CD be not the ſame with the ratio of AE to CF, it is either greater than it, or, by inverſio the ratio of CD to AB is greater than the ratio of CF to A Firſt, let the ratio of AB to CD be greater than the ratio AE to CF; and as AE to CF, ſo make AG to CD; there fore the ratio of AG to CD is given, becauſe the ratio

b 2. dat. AE to to CF is given; and CD is given, wherefore b AG given

given; and becauſe the ratio of AB to CD is greater than the ratio of (AE to CF, that is, than the ratio of) AG to CD; AB is greater [c] than AG: And AB, AG are given; therefore the remainder BG is given: And becauſe as AE to CF, ſo is AG to CD, and ſo is [a] EG to FD; the ratio of EG to FD is given: And GB is given; therefore EG, the exceſs of EB above a given magnitude GB, has a given ratio to FD. The other caſe is ſhewn in the ſame way.



[c] 10. 5.

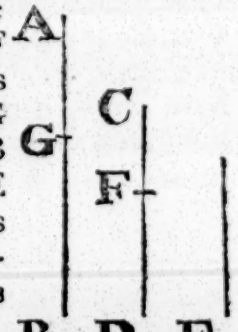
[a] 19. 5.

PROP. XXIV. 13.

IF there be three magnitudes, the firſt of which has a given ratio to the ſecond, and the exceſs of the ſecond above a given magnitude has a given ratio to the third; the exceſs of the firſt above a given magnitude ſhall alſo have a given ratio to the third. See N.

Let AB, CD, E, be the three magnitudes of which AB has a given ratio to CD; and the exceſs of CD above a given magnitude has a given ratio to E: The exceſs of AB above a given magnitude has a given ratio to E.

Let CF be the given magnitude, the exceſs of CD above which, viz. FD has a given ratio to E: And becauſe the ratio of AB to CD is given, as AB to CD, ſo make AG to CF; therefore the ratio of AG to CF is given; and CF is given, wherefore [a] AG is given: And becauſe as AB to CD, ſo is AG to CF, and ſo is [b] GB to FD: the ratio of GB to FD is given. And the ratio of FD to E given, wherefore [c] the ratio of GB to E is given, and AG is given; therefore GB the exceſs of AB above a given magnitude AG has a given ratio to E.



[a] 2. dat.

[b] 19. 5.

[c] 9. dat.

Cor. 1. And if the firſt has a given ratio to the ſecond, and the exceſs of the firſt above a given magnitude has a given ratio to the third; the exceſs of the ſecond above a given magnitude ſhall have a given ratio to the third. For, if the ſecond be called the firſt, and the firſt the ſecond, this corollary will be the ſame with the propoſition.

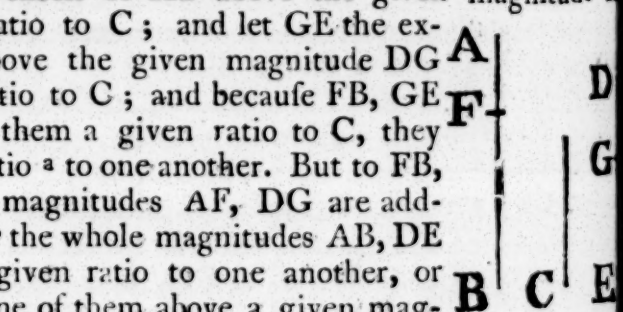
Cor. 2. Alſo if the firſt has a given ratio to the ſecon and the exceſs of the third above a given magnitude has alſo given ratio to the ſecond, the ſame exceſs ſhall have a giv ratio to the firſt; as is evident from the 9th dat.

17. PROP. XXV.

IF there be three magnitudes, the exceſs of the fir whereof above a given magnitude has a given rat to the ſecond; and the exceſs of the third above given magnitude has a given ratio to the ſame ſ cond: The firſt ſhall either have a given ratio to t third, or the exceſs of one of them above a give magnitude ſhall have a given ratio to the other.

Let AB, C, DE be three magnitudes, and let the exceſ of each of the two AB, DE above given magnitudes have giv ratios to C; AB, DE either have a given ratio to one anoth or the exceſs of one of them above a given magnitude ha given ratio to the other.

Let FB the exceſs of AB above the given magnitude A have a given ratio to C; and let GE the exceſs of DE above the given magnitude DG have a given ratio to C; and becauſe FB, GE have each of them a given ratio to C, they have a given ratio [a] to one another. But to FB, GE the given magnitudes AF, DG are added; therefore [b] the whole magnitudes AB, DE have either a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.



a 9. dat.

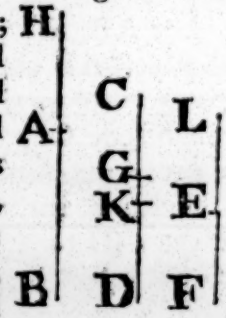
b 18. dat.

18. PROP. XXVI.

IF there be three magnitudes, the exceſſes of o of which above given magnitudes have given r tios to the other two magnitudes; theſe two ſh either have a given ratio to one another, or the e ceſs of them above a given magnitude ſhall ha a given ratio to the other.

Let AB, CD, EF be three magnitudes, and let GD the exceſs of one of them CD above the given magnitude CG have a given ratio to AB; and alſo let KD the exceſs of the ſame CD above the given magnitude CK have a given ratio to EF, either AB has a given ratio to EF, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Becauſe GD has a given ratio to AB, as GD to AB, ſo make CG to HA; therefore the ratio of CG to HA is given; and CG is given, wherefore [a] HA is given: And becauſe as GD to AB, ſo is CG to HA, and ſo is [b] CD to HB; the ratio of CD to HB is given: Alſo becauſe KD has a given ratio to EF, as KD to EF, ſo make CK to LE; therefore the ratio of CK to LE is given; and CK is given, wherefore LE [a] is given: And becauſe as KD to EF, ſo is CK to LE, and ſo [b] is CD to LF; the ratio of CD to LF is given: But the ratio of CD to HB is given, wherefore [c] the ratio of HB to LF is given: and from HB, LF the given magnitudes HA, LE being taken, the remainders AB, EF ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other [d].

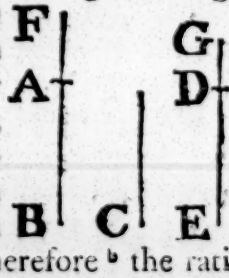


[a] 2. Dat. [b] 12. 5. [c] 9. Dat. [d] 19. Dat.

"Another demonſtration.

Let AB, C, DE be three magnitudes, and let the exceſſes of one of them C above given magnitudes have given ratios to AB and DE; either AB, DE have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Becauſe the exceſs of C above a given magnitude has a given ratio to AB; therefore [a] AB together with a given magnitude has a given ratio to C: Let this given magnitude be AF, wherefore FB has a given ratio to C: Alſo, becauſe the exceſs of C above a given magnitude has a given ratio to DE; therefore [a] DE together with a given magnitude has a given ratio to C: Let this given magnitude be DG, wherefore GE has a given ratio to C: And FB has a given ratio to C, therefore [b] the ratio of FB to GE is given: And from FB, GE the given magnitudes AF, DG being taken, the remainders AB, DE either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other [c]."



[a] 14. Dat. [b] 9. Dat. [c] 19. Dat.

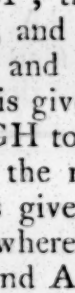
PROP.

19. PROP. XXVII.

IF there be three magnitudes: the excefs of the firſt of which above a given magnitude has a given ratio to the ſecond; and the excefs of the ſecond above a given magnitude has alſo a given ratio to the third: The excefs of the firſt above a given magnitude ſhall have a given ratio to the third.

Let AB, CD, E be three magnitudes, the excefs of the firſt of which AB above the given magnitude AG, viz. GB, has a given ratio to CD; and FD the excefs of CD above the given magnitude CF, has a given ratio to E: The excefs of AB above a given magnitude has a given ratio to E.

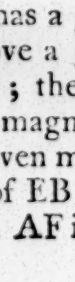
Becauſe the ratio of GB to CD is given, as GB to CD, ſo make GH to CF; therefore the ratio of GH
[a] 2. Dat. to CF is given; and CF is given, wherefore [a] GH is given; and AG is given, wherefore the whole AH is given: And becauſe as GB
[b] 19. 5. to CD, ſo is GH to CF, and ſo is [b] the remainder HB to the remainder FD; the ratio of HB to FD is given: And the ratio of FD
[c] 9. Dat. to E is given, wherefore [c] the ratio of HB to E is given: And AH is given; therefore HB the excefs of AB above a given magnitude AH has a given ratio to E.



" Otherwiſe.

Let AB, C, D be three magnitudes, the excefs EB of the firſt of which AB above the given magnitude AE has a given ratio to C, and the excefs of C above a given magnitude has a given ratio to D: The excefs of AB above a given magnitude has a given ratio to D.

Becauſe EB has a given ratio to C, and the excefs of C above a given magnitude has a gi-
[d] 24. Dat. ven ratio to D; therefore [d] the excefs of EB above a given magnitude has a given ratio to D: Let this given magnitude be EF; therefore FB the excefs of EB above EF has a given ratio to D: And AF is given, becauſe AE, EF

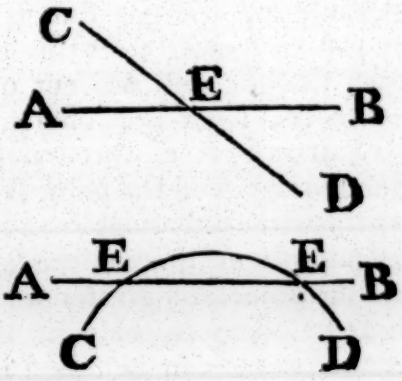


re given: Therefore FB the excess of AB above a given mag-
itude AF has a given ratio to D."

PROP. XXVIII. 25.

F two lines given in position cut one another, the point or points in which they cut one another are given. See N.

Let two lines AB, CD given in position cut one another in he point E; the point E is gi-
en.

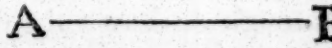


Becauſe the lines AB, CD re given in position, they have lways the ſame ſituation [a], and herefore the point, or points n which they cut one another ave always the ſame ſituation: And becauſe the lines AB, CD an be found [a], the point, or oints, in which they cut one nother, are likewiſe found; nd therefore are given in poſition [a]. [a] 4. Def.

PROP. XXIX. 26.

F the extremities of a ſtraight line be given in po-
ſition; the ſtraight line is given in poſition and magnitude.

Becauſe the extremities of the ſtraight line are given, they an be found [a]: let theſe be the points A, B, between which ſtraight line AB can be drawn [b]; his has an invariable poſition, be-
auſe between two given points there
an be drawn but one ſtraight line: And when the ſtraight line AB is drawn, its magnitude is at the ſame time exhibited, or iven: Therefore the ſtraight line AB is given in poſition and agnitude.



[a] 4. Def.
[b] 1. Poſtulate.

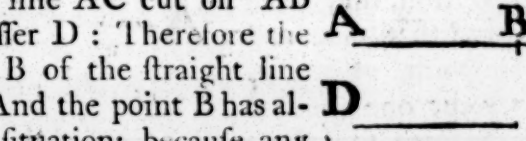
PROP.

27. PROP. XXX.

IF one of the extremities of a ſtraight line given in poſition and magnitude be given; the other extremity ſhall alſo be given.

Let the point A be given, to wit, one of the extremities of a ſtraight line given in magnitude, and which lies in the ſtraight line AC given in poſition; the other extremity is alſo given.

Becauſe the ſtraight line is given in magnitude, one equal to it can be found [a]; let this be the ſtraight line D. From the greater ſtraight line AC cut off AB equal to the leſſer D: Therefore the other extremity B of the ſtraight line AB is found: And the point B has always the ſame ſituation; becauſe any other point in AC, upon the ſame ſide of A, cuts off between it and the point A a greater or leſs ſtraight line than AB, that is than D: Therefore the point B is given [b]: And it is plain another ſuch point can be found in AC produced upon the other ſide of the point A.



[a] 1. def.

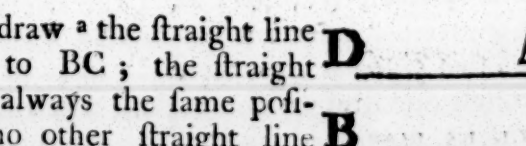
[b] 4. def.

28. PROP. XXXI.

IF a ſtraight line be drawn through a given point parallel to a ſtraight line given in poſition; that ſtraight line is given in poſition.

Let A be a given point, and BC a ſtraight line given in poſition; the ſtraight line drawn through A parallel to BC is given in poſition.

Through A draw [a] the ſtraight line DAE parallel to BC; the ſtraight line DAE has always the ſame poſition, becauſe no other ſtraight line can be drawn through A parallel to BC: Therefore the ſtraight line DAE which has been found is given [b] in poſition.



[a] 31. 1.

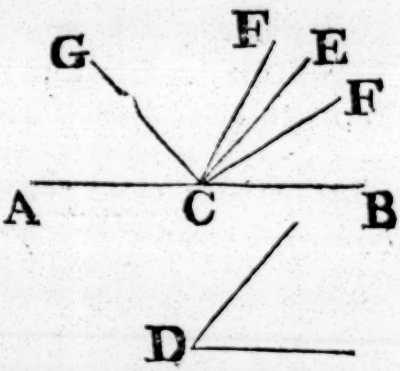
[b] 4. def.

PROP

PROP. XXXII. 29.

IF a ſtraight line be drawn to a given point in a ſtraight line given in poſition, and makes a given angle with it; that ſtraight line is given in poſition.

Let AB be a ſtraight line given in poſition, and C a given point in it, the ſtraight line drawn to C which makes a given angle with CB, is given in poſition.



Becauſe the angle is given, one equal to it can be found [a]; let this be the angle at D, at the given point C in the given ſtraight line AB make [b] the angle ECB equal to the angle at D: Therefore the ſtraight line EC has always the ſame ſituation, becauſe any other ſtraight line FC drawn to the point C makes with CB a greater or leſs angle than the angle ECB or the angle at D: Therefore the ſtraight line EC, which has been found, is given in poſition.

[a] 1. Def.

[b] 23. 1.

It is to be obſerved, that there are two ſtraight lines EC, GC upon one ſide of AB that make equal angles with it, and which make equal angles with it when produced to the other ſide.

PROP. XXXIII. 30.

IF a ſtraight line be drawn from a given point, to a ſtraight line given in poſition, and makes a given angle with it; that ſtraight line is given in poſition.

From the given point A, let the ſtraight line AD be drawn to the ſtraight line BC given in poſition, and make with it a given angle ADC; AD is given in poſition.

Thro' the point A, draw [a] the ſtraight line EAF parallel to BC; and becauſe thro' the given point A the ſtraight line EAF is drawn parallel to BC which is given in poſition, EAF is therefore given in poſition [b]: And becauſe the ſtraight line AD meets the parallels BC, EF, the angle

[a] 31. 1.

[b] 31. Dat.

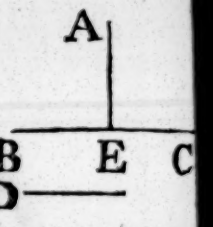
c 29. 1. angle EAD is equal c to the angle ADC; and ADC is give
wherefore alſo the angle EAD is given: Therefore, becauſe t
ſtraight line DA is drawn to the given point A in the ſtraig
line EF given in poſition, and makes with it a given ang
d 32. Dat. EAD, AD is given d in poſition.

31. PROP. XXXIV.

See N. IF from a given point to a ſtraight line given i poſition, a ſtraight line be drawn which is give in magnitude; the ſame is alſo given in poſition.

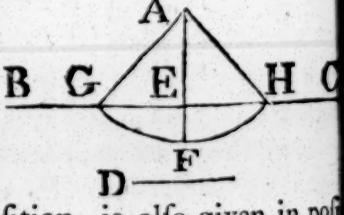
Let A be a given point, and BC a ſtraight line given poſition, a ſtraight line given in magnitude drawn from t point A to BC is given in poſition.

Becauſe the ſtraight line is given in magnitude, one equ
a 1. def. to it can be found a; let this be the ſtraight line D: From t
point A draw AE perpendicular to BC; and becauſe AE is the ſhorteſt of all the ſtraight lines which can be drawn from the point A to BC, the ſtraight line D, to which one equal is to be drawn from the point A to BC, cannot be leſs than AE. If therefore D be equal to AE, AE is the ſtraight line given in magnitude drawn from the given point A to BC



b 33. Dat. And it is evident that AE is given in poſition b becauſe it drawn from the given point A to BC which is given in poſition and makes with BC the given angle AEC.

But if the ſtraight line D be not equal to AE, it muſt b greater than it: Produce AE, and make AF equal to D; an from the center A, at the diſtance AF, deſcribe the circ GFH, and join AG, AH: Becauſe the circle GFH is give
c 6. def. in poſition c, and the ſtraight line BC is alſo given in poſition therefore their interſection G is gi-
d 28. Dat. ven d; and the point A is given;
e 29. Dat. wherefore AG is given in poſition e, that is, the ſtraight line AG given in magnitude (for it is equal to D) and drawn from the given point A to the ſtraight line BC given in poſition, is alſo given in poſi tion: And in like manner AH is given in poſition: Therefore in this caſe there are two ſtraight lines AG, AH of the ſam



give

iven magnitude which can be drawn from a given point A to ſtraight line BC given in poſition.

PROP. XXXV. 32.

F a ſtraight line be drawn between two parallel ſtraight lines given in poſition, and makes given ngles with them; the ſtraight line is given in mag-itude.

Let the ſtraight line EF be drawn between the parallels AB, D which are given in poſition, and make the given angles EF, EFD; EF is given in magnitude.

In CD take the given point G, and through G draw [a] GH parallel to EF: And becauſe CD meets the parallels GH, EF, e angle EFD is equal [b] to the angle GD: And EFD is a given angle; herefore the angle HGD is given: And ecauſe HG is drawn to the given point in the ſtraight line CD given in po-ion, and makes a given angle HGD; e ſtraight line HG is given in poſi-on [c]: And AB is given in poſition; therefore the point H is iven [d]; and the point G is alſo given, wherefore GH is given n magnitude [e]: And EF is equal to it; therefore EF is given n magnitude.

[a] 31. 1.
[b] 29. 1.
[c] 32. Dat.
[d] 28. Dat.
[e] 29. Dat.

PROP. XXXVI. 33.

F a ſtraight line given in magnitude be drawn be- See N. tween two parallel ſtraight lines given in poſition; ſhall make given angles with the parallels.

Let the ſtraight line EF given in magnitude be drawn be-ween the parallel ſtraight lines AB, CD hich are given in poſition; the angles EF, EFC ſhall be given.

Becauſe EF is given in magnitude, a raight line equal to it can be found [a]; t this be G: In AB take a given point , and from it draw [b] HK perpendicu-r to CD: Therefore the ſtraight line G,

[a] 1. def.
[b] 12. 1.

that

that is, EF, cannot be leſs than HK: And if G be equal to HK, EF alſo is equal to it; wherefore EF is at right angles to CD; for if it be not, EF would be greater than HK, which is abſurd. Therefore the angle EFD is a right, and conſequently a given angle.

But if the ſtraight line G be not equal to HK, it muſt be greater than it: Produce HK, and take HL, equal to G; and from the center H, at the diſtance HL, deſcribe the circle [c] MLN, and join HM, HN: And becauſe the circle [c] MLN, and the ſtraight line CD are given in poſition, the points M, N are [d] given; and the point H is given; wherefore the ſtraight lines HM, HN are given in poſition [e]: And CD is given in poſition; therefore the angles HMN, HNM are given in poſition [f]: Of the ſtraight lines HM, HN, let HN be that which is not parallel to EF, for EF cannot be parallel to both of them; and draw EO parallel to HN: EO therefore is equal [g] to HN, that is, to G; and EF is equal to G; wherefore EO is equal to EF, and the angle EFO to the angle EOF, that is [h], to the given angle HNM, and becauſe the angle HNM which is equal to the angle EFO or EFD has been found, therefore the angle EFD, that is, the angle AEF, is given in magnitude [k]; and conſequently the angle EFC.

[c] 6. def.
[d] 28. dat.
[e] 29. dat.
[f] A. def.
[g] 34. 1.
[h] 29. 1.
[k] 1. def.

E. See N.

PROP. XXXVII.

IF a ſtraight line given in magnitude be drawn from a point to a ſtraight line given in poſition in a given angle; the ſtraight line drawn through that point parallel to the ſtraight line given in poſition, is given in poſition.

Let the ſtraight line AD given in magnitude be drawn from the point A to the ſtraight line BC given in poſition, in the given angle ADC; the ſtraight line EAF drawn through A parallel to BC is given in poſition.

In BC take a given point G, and draw GH parallel to AD: And becauſe HG is drawn to a given point G in the ſtraight line BC gi-

n in position, in a given angle HGC, for it is equal[a] to the a. 29. 1.
ven angle ADC; HG is given in position[b]; but it is given b. 32. Dat.
o in magnitude, becauſe it is equal to[c] AD which is given c. 34. 1.
magnitude; therefore becauſe G one of the extremities of
e ſtraight line GH given in poſition and magnitude is given,
e other extremity H is given[d]; and the ſtraight line EAF d. 30. Dat.
hich is drawn thro' the given point H parallel to BC given in
ſition, is therefore given[e] in poſition. e. 31. Dat.

PROP. XXXVIII. 34.

F a ſtraight line be drawn from a given point to two parallel ſtraight lines given in poſition; the ratio of e ſegments between the given point and the parallels all be given.

Let the ſtraight line EFG be drawn from the given point to the parallels AB, CD; the ratio of EF to EG is given.
From the point E draw EHK perpendicular to CD; and ecauſe from a given point E the ſtraight line EK is drawn to D which is given in poſition, in a given angle EKC; EK is

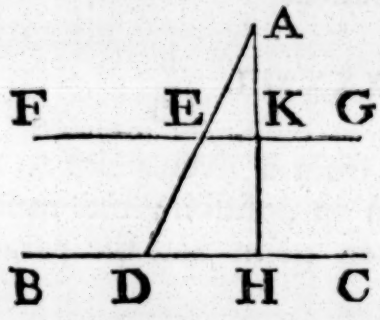
ven in poſition[a]; and AB, CD are given in poſition; there- a. 33. Dat.
re[b] the points H, K are given: And the point E is given; b. 28. Dat.
herefore[c] EH, EK are given in magnitude, and the ratio[d] c. 29. Dat.
f them is therefore given. But as EH to EK, ſo is EF to EG, d. 1. Dat.
ecauſe AB, CD are parallels; therefore the ratio of EF to G is given.

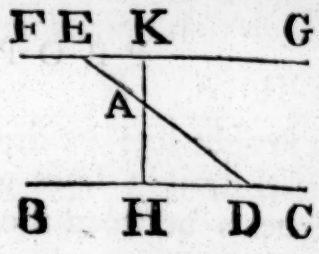
PROP. XXXIX. 35. 36.

F the ratio of the ſegments of a ſtraight line between a given point in it and two parallel ſtraight lines, be ven; if one of the parallels be given in poſition, the her is alſo given in poſition. See N.

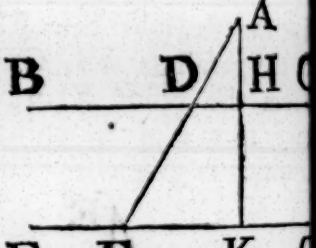
From the given point A, let the ſtraight line AED be draw to the two parallel ſtraight lines FG, BC, and let the ratio the ſegments AE, AD be given; if one of the parallels B be given in poſition, the other FG is alſo given in poſition.

From the point A, draw AH perpendicular to BC, and l it meet FG in K; and becauſe AH is drawn from the give point A to the ſtraight line BC given in poſition, and makes





a.33.Dat. given angle AHD; AH is given[a] in poſition; and BC is likewiſe given in poſition, therefore the point H is gi-
b.28.Dat. ven[b]: The point A is alſo given;
c.29.Dat. wherefore AH is given in magnitude[c]. and, becauſe FG, BC are parallels, as AE to AD, ſo is AK to AH; and the ratio of AE to AD is given, wherefore the ratio of AK to AH is given; but AH is give
d.2.Dat. in magnitude, therefore[d] AK is given in magnitude; an it is alſo given in poſition, and the point A is given; where
e.30.Dat. fore[e] the point K is given. And becauſe the ſtraight line F is drawn thro' the given point K parallel to BC which
f.31.Dat. given in poſition, therefore[f] FG is given in poſition.



37. 38. PROP. XL.

See N. IF the ratio of the ſegments of a ſtraight line into whic it is cut by three parallel ſtraight lines, be given; two of the parallels are given in poſition, the third al is given in poſition.

Let AB, CD, HK be three parallel ſtraight lines, of whic AB, CD are given in poſition; and let the ratio of the ſe
men

ents GE, GF into which the ſtraight line GEF is cut by the ree parallels, be given; the third parallel HK is given in ſition.

In AB take a given point L, and draw LM perpendicular CD, meeting HK in N; becauſe LM is drawn from the ven point L to CD which is given in poſition and makes a ven angle LMD; LM is given in poſition [a]; and CD is gi- a. 33. Dat.
n in poſition, wherefore the point M is given [b]; and the b. 28. Dat.
int L is given, LM is therefore given in magnitude [c]; and c. 29. Dat.
ecauſe the ratio of GE to GF is given, and as GE to GF, ſo

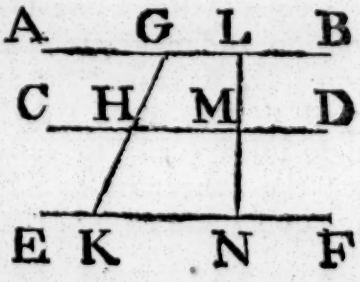
NL to NM; the ratio of NL to NM is given; and there- re [d] the ratio of ML to LN is given; but LM is given in agnitude, wherefore [e] LN is given in magnitude; and it is ſo given in poſition, and the point L is given; wherefore [f] e point N is given; and becauſe the ſtraight line HK is awn thro' the given point N parallel to CD which is given in ſition, therefore HK is given in poſition [g]. d. Cor. 6. or 7 Dat. e. 2. Dat. f. 30. Dat. g. 31. Dat.

PROP. XLI. F.

F a ſtraight line meets three parallel ſtraight lines which are given in poſition; the ſegments into which ey cut it, have a given ratio. See N.

Let the parallel ſtraight lines AB, CD, EF given in poſi- on be cut by the ſtraight line GHK; the ratio of GH to K is given.

In AB take a given point L, and aw LM perpendicular to CD, meet- g EF in N; therefore [a] LM is given poſition; and CD, EF are given poſition, wherefore the points M, are given: And the point L is given; erefore [b] the ſtraight lines LM, MN e given in magnitude; and the ratio



a 33. Dat.

b 29. Dat.

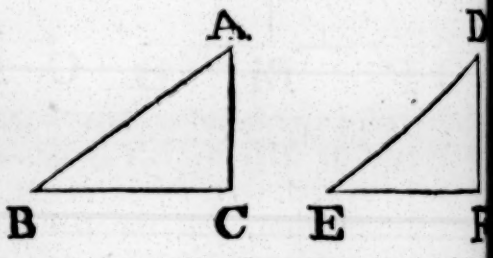
c. 1. Dat. of LM to MN is therefore given[c]: But as LM to MN, so GH to HK; wherefore the ratio of GH to HK is given.

39. PROP. XLII.

See N. IF each of the ſides of a triangle be given in magnitude the triangle is given in ſpecies.

Let each of the ſides of the triangle ABC be given in magnitude; the triangle ABC is given in ſpecies.

a. 22. 1. Make a triangle[a] DEF the ſides of which are equal, each to each, to the given ſtraight lines AB, BC, CA; which can be done, becauſe any two of them muſt be greater than the third; and let DE be equal to AB, EF to BC, and FD to CA; and becauſe the two ſides ED, DF are equal to the two BA, AC, each to each, and the baſe EF equal to the baſe BC; the angle



b. 8. 1. EDF is equal[b] to the angle BAC; therefore, becauſe the angle EDF, which is equal to the angle BAC, has been found, the

c. 1. Def. angle BAC is given[c], in like manner the angles at B, C are given. And becauſe the ſides AB, BC, CA are given, the

d. 1. Dat. ratios to one another are given[d]. therefore the triangle AB

e. 3. Def. is given[e] in ſpecies.

40. PROP. XLIII.

IF each of the angles of a triangle be given in magnitude; the triangle is given in ſpecies.

Let each of the angles of the triangle ABC be given in magnitude; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude, and at the

a. 23. 1. points D, E make[a] the angle EDF equal to the angle BAC, and the angle DEF equal to ABC; therefore the other angles EFD, BCA are equal; and each of the angles at the points A, B, C, is

ven, wherefore each of thoſe at the points D, E, F is given: And becauſe the ſtraight line FD is drawn to the given point D in DE which is given in poſition, making the given angle EDF; therefore DF is given in poſition [b]. In like manner EF alſo is given in poſition; wherefore the point F is given: And the points D, E are given; therefore each of the ſtraight lines DE, EF, FD is given [c] in magnitude; wherefore the triangle DEF is given in ſpecies [d]; and it is ſimilar [e] to the triangle ABC; which therefore is given in ſpecies.

b. 32. Dat.

c. 29. Dat.

d. 42. Dat.

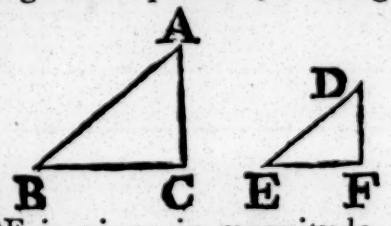
e { 4. 6. / 1. Def. 6. / 41.

PROP. XLIV.

IF one of the angles of a triangle be given, and if the ſides about it have a given ratio to one another; the triangle is given in ſpecies.

Let the triangle ABC have one of its angles BAC given, and let the ſides BA, AC about it have a given ratio to one another; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude, and at the point D in the given ſtraight line DE, make the angle EDF equal to the given angle BAC; wherefore the angle EDF is given; and becauſe the ſtraight line FD is drawn to the given point D in ED which is given in poſition, making the given angle EDF; therefore FD is given in poſition [a]. And becauſe the ratio of BA to AC is given, make the ratio of ED to DF the ſame with it, and join EF; and becauſe the ratio of ED to DF is given, and ED is given, therefore [b] DF is given in magnitude; and it is given alſo in poſition, and the point D is given, wherefore the point F is given [c]; and the points D, E are given, wherefore DE, EF, FD are given [d] in magnitude; and the triangle DEF is therefore given [e] in ſpecies; and becauſe the triangles ABC, DEF have one angle BAC equal to one angle EDF, and the ſides about theſe angles proportionals; the triangles are [f] ſimilar; but the triangle DEF is given in ſpecies, and therefore alſo the triangle ABC.



a. 32. Dat.

b. 2. Dat.

c. 30. Dat.

d. 29. Dat.

e. 42. Dat.

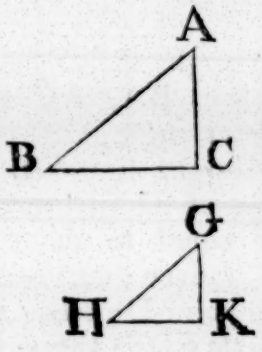
f. 6. 6.

42. PROP. XLV.

See N. IF the ſides of a triangle have to one another given ra-
tios; the triangle is given in ſpecies.

Let the ſides of the triangle ABC have given ratios to o
another, the triangle ABC is given in ſpecies.

Take a ſtraight line D given in magnitude; and becauſe t
ratio of AB to BC is given, make the ratio of D to E t
a. 2. Dat. ſame with it; and D is given, therefore [a] E is given. And b
cauſe the ratio of BC to CA is given, to this make the ra
of E to F the ſame; and E is given, and therefore [a] F; a
becauſe as AB to BC, ſo is D to E, by compoſition AB a
BC together are to BC, as D and E to E; but as BC to CA, ſo is E to F; therefore, ex aequali
b. 22. 5. [b], as AB and BC are to CA, ſo are D and E to F, and
c. 20. 1. AB and BC are greater [c] than CA; therefore D and E are
d. A. 5. greater [d] than F. In the ſame manner any two of the three D, E, F are greater than the third.



e. 22. 1. Make [e] the triangle GHK whoſe ſides are equal to D, E, F, ſo that GH be equal to D, HK
E, and KG to F; and becauſe D, E, F are, each of the
given, therefore GH, HK, KG are each of them given in ma
f. 42. Dat. nitude; therefore the triangle GHK is given [f] in ſpecies: B
as AB to BC, ſo is (D to E, that is) GH to HK; and as BC
CA, ſo is (E to F, that is) HK to KG; therefore, ex aequ
g. 5. 6. as AB to AC, ſo is GH to GK. Wherefore [g] the triangle A
is equiangular and ſimilar to the triangle GHK; and the
angle GHK is given in ſpecies; therefore alſo the triangle A
is given in ſpecies.

COR. If a triangle is required to be made the ſides of wh
ſhall have the ſame ratios which three given ſtraight lines
E, F have to one another; it is neceſſary that every two
them be greater than the third.

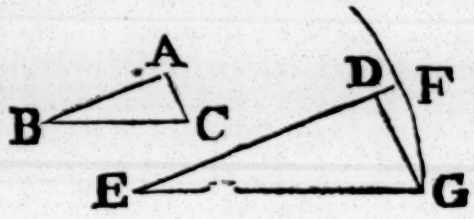
PRO

PROP. XLVI. 43.

IF the ſides of a right angled triangle about one of the acute angles have a given ratio to one another; the triangle is given in ſpecies.

Let the ſides AB, BC about the acute angle ABC of the triangle ABC, which has a right angle at A, have a given ratio to one another; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude; and becauſe the ratio of AB to BC is given, make as AB to BC, ſo DE to EF; and becauſe DE has a given ratio to EF, and DE is given, therefore [a] EF is given; and becauſe as AB to BC, ſo is DE to EF; and AB is leſs [b] than BC, therefore DE is leſs [c] than EF. From the point D draw DG at right angles to DE, and from the center E at the diſtance EF deſcribe a circle which ſhall meet DG in two points; let G be either of them, and join EG; therefore the circumference of the circle is given [d] in poſition;



and the ſtraight line DG is given [e] in poſition, becauſe it is drawn to the given point D in DE given in poſition, in a given angle; therefore [f] the point G is given; and the points D, E are given, wherefore DE, EG, GD are given [g] in magnitude, and the triangle DEG in ſpecies [h]. And becauſe the triangles ABC, DEG have the angle BAC equal to the angle EDG, and the ſides about the angles ABC, DEG proportionals, and each of the other angles BCA, EGD leſs than a right angle; the triangle ABC is equiangular [i] and ſimilar to the triangle DEG: But DEG is given in ſpecies; therefore the triangle ABC is given in ſpecies. And in the ſame manner, the triangle made by drawing a ſtraight line from E to the other point in which the circle meets DG is given in ſpecies.

a. 2. Dat.
b. 19. 1.
c. A. 5.
d. 6. Def.
e. 32. Dat.
f. 28. Dat.
g. 29. Dat.
h. 42. Dat.
i. 7. 6.

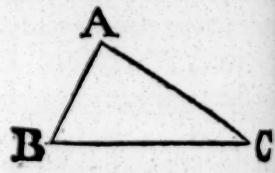
44. PROP. XLVII.

See N. IF a triangle has one of its angles which is not a rig angle given, and if the ſides about another angle ha a given ratio to one another; the triangle is given ſpecies.

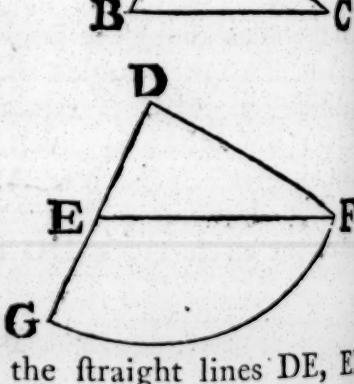
Let the triangle ABC have one of its angles ABC a give but not a right angle, and let the ſides BA, AC about an other angle BAC have a given ratio to one another; the tr angle ABC is given in ſpecies.

Firſt, Let the given ratio be the ratio of equality, that is, let the ſides BA, AC, and conſequently the angles ABC, ACB, be equal; and becauſe the angle ABC is given, the
a. 32. 1. angle ACB, and alſo the remaining [a] angle BAC is given; therefore the triangle ABC is
b. 43. Dat. given [b] in ſpecies: And it is evident that in this caſe the given angle ABC muſt be acute.

Next, Let the given ratio be the ratio of a leſs to a greate that is, let the ſide AB adjacent to the given angle be le than the ſide AC: Take a ſtraight line DE given in poſitio and magnitude, and make the angle DEF equal to the give
c. 32. Dat. angle ABC; therefore EF is given [c] in poſition; and becau

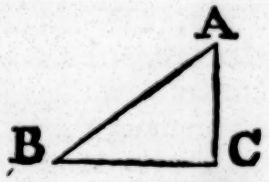


the ratio of BA to AC is given, as BA to AC, ſo make ED to DG; and becauſe the ratio of ED to DG is given, and ED is given, the ſtraight line DG is gi-
d. 2. Dat. ven [d], and BA is leſs than AC,
e. A. 5. therefore ED is leſs [e] than DG. From the center D, at the diſtance DG deſcribe the circle GF meeting EF in F, and join DF; and
f. 6. Def. becauſe the circle is given [f] in poſition, as alſo the ſtraight line EF,
g. 28. Dat. the point F is given [g]; and the points D, E are given, wherefore the ſtraight lines DE, EF
h. 29 Dat. FD are given [h] in magnitude, and the triangle DEF in ſpe
i. 42. Dat. cies [i]. and becauſe BA is leſs than AC, the angle ACB i
k. 18. 1. leſs [k] than the angle ABC, and therefore ACB is leſs [l] tha
l 1. 7. 1. a righ

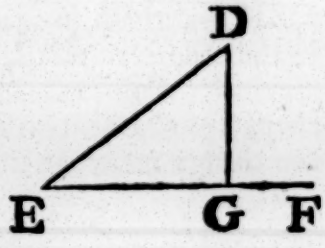


a right angle. In the ſame manner, becauſe ED is leſs than DG or DF, the angle DFE is leſs than a right angle: And becauſe the triangles ABC, DEF have the angle ABC equal to the angle DEF, and the ſides about the angles BAC, EDF proportionals, and each of the other angles ACB, DFE leſs than a right angle; the triangles ABC, DEF are [m] ſimilar, and DEF is given in ſpecies, wherefore the triangle ABC is alſo given in ſpecies. m. 7. 6.

Thirdly, Let the given ratio be the ratio of a greater to a leſs, that is, let the ſide AB adjacent to the given angle be greater than AC; and, as in the laſt caſe, take a ſtraight line DE given in poſition and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given [c] in poſition: Alſo draw DG perpendicular to EF; therefore if the ratio of BA to AC be the ſame with the ratio of ED to the perpendicular DG, the triangles ABC, DEG are ſimilar [m], becauſe the angles ABC, DEG are equal, and DGE is a right angle: Therefore the angle ACB is a right angle, and the triangle ABC is given [b] in ſpecies.



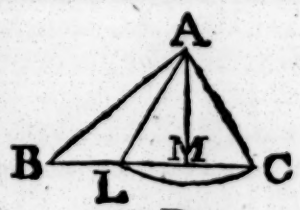
c. 32. Dat.



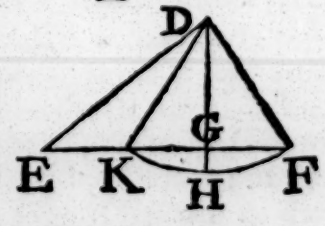
b. 43. Dat

But if, in this laſt caſe, the given ratio of BA to AC be not the ſame with the ratio of ED to DG, that is, with the ratio of BA to the perpendicular AM drawn from A to BC; the ratio of BA to AC muſt be leſs [o] than the ratio of BA to AM, becauſe AC is greater than AM. Make as BA to AC ſo ED to DH; therefore the ratio of ED to DH is leſs than the ratio of (BA to AM, that is, than the ratio of) ED to DG; and conſequently, DH is greater [p] than DG; and becauſe BA is greater than AC, ED is greater [e] than DH. From the center D, at the diſtance DH, deſcribe the circle KHF which neceſſarily meets the ſtraight line EF in two points, becauſe DH is greater than DG, and leſs than DE. Let the circle meet EF in the points F, K which are given, as was ſhewn in the preceeding caſe; and, DF, DK being joined, the triangles DEF, DEK are given in ſpecies, as was there ſhewn.

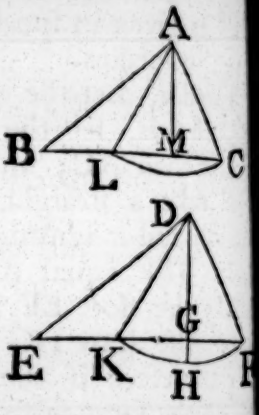
o. 8. 5.



p. 10. 5.
e. A. 5.



ſhewn. From the center A, at the diſtance AC, deſcribe a c
cle meeting BC again in L: And if the angle ACB be l
than a right angle, ALB muſt be greater than a right angl
And on the contrary. In the ſame manner, if the angle DFE
leſs than a right angle, DKE muſt be greater than one; a
on the contrary. Let each of the angles ACB, DFE be either leſs or greater than a right angle; and becauſe in the triangles ABC, DEF the angles ABC, DEF are equal, and the ſides BA, AC, and ED, DF about two of the other angles proportionals, the triangle ABC
m. 7. 6. is ſimilar [m] to the triangle DEF. In the ſame manner, the triangle ABL is ſimilar to DEK. And the triangles, DEF, DEK are given in ſpecies; therefore alſo the triangles ABC, ABL are given in ſpecies. And from this it is evident, that, in this third ca
there are always two triangles of a different ſpecies, to whi
the things mentioned as given in the propoſition can agree.

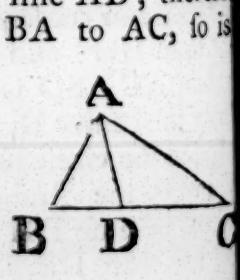


45. PROP. XLVIII.

IF a triangle has one angle given, and if both the ſid
together about that angle have a given ratio to th
remaining ſide; the triangle is given in ſpecies.

Let the triangle ABC have the angle BAC given, and let th
ſides BA, AC together about that angle have a given ratio
BC; the triangle ABC is given in ſpecies.

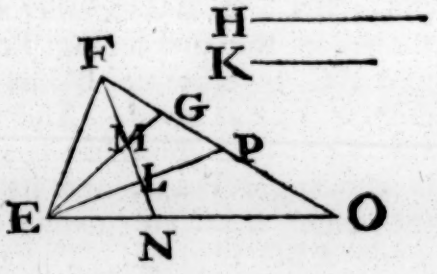
a. 9. 1. Biſect [a] the angle BAC by the ſtraight line AD; therefo
b. 3. 6. the angle BAD is given. And becauſe as BA to AC, ſo is BD to DC, by permutation, as AB to BD, ſo is AC to CD; and as BA and
c. 12. 5. AC together to BC, ſo is [c] AB to BD. But the ratio of BA and AC together to BC is given, wherefore the ratio of AB to BD is given, and the angle BAD is
d. 47. Dat. given; therefore [d] the triangle ABD is given in ſpecies, and the angle ABD is therefore given; the angl
e. 43. Dat. BAC is alſo given, wherefore the triangle ABC is given in ſpecies.



A triangle which ſhall have the things that are mentione
in the Propoſition to be given, can be found in the followin
manne

manner. Let EFG be the given angle, and let the ratio of H to K be the given ratio which the two ſides about the angle EFG muſt have to the third ſide of the triangle; therefore becauſe two ſides of a triangle are greater than the third ſide, the ratio of H to K muſt be the ratio of a greater to a leſs. Biſect [a] the angle EFG by the ſtraight line FL, and by the 47th Propoſition find a triangle of which EFL is one of the angles, and in which the ratio of the ſides about the angle oppoſite to FL is the ſame with the ratio of H to K: To do which, take FE given in poſition and magnitude, and draw EL perpendicular to FL: Then, if the ratio of H to K be the ſame with the ratio of FE to EL, produce EL, and let it meet FG in P; the triangle FEP is that which was to be found: For it has the given angle EFG; and becauſe this angle is biſected by FL, the ſides EF, FP together are to EP, as [b] FE to EL, that is, as H to K.

a. 9. 1.



b. 3. 6.

But if the ratio of H to K be not the ſame with the ratio of FE to EL, it muſt be leſs than it, as was ſhewn in Prop. 47. and in this caſe there are two triangles, each of which has the given angle EFL, and the ratio of the ſides about the angle oppoſite to FL the ſame with the ratio of H to K. By Prop. 47. find theſe triangles EFM, EFN each of which has the angle EFL for one of its angles, and the ratio of the ſide FE to EM or EN the ſame with the ratio of H to K; and let the angle EMF be greater, and ENF leſs than a right angle. And becauſe H is greater than K, EF is greater than EN, and therefore the angle EFN, that is, the angle NFG, is leſs [f] than the angle ENF. To each of theſe add the angles NEF, EFN; therefore the angles NEF, EFG are leſs than the angles NEF, EFN, FNE, that is, than two right angles; therefore the ſtraight lines EN, FG muſt meet together when produced; let them meet in O, and produce EM to G. Each of the triangles, EFG, EFO has the things mentioned to be given in the Propoſition: For each of them has the given angle EFG; and becauſe this angle is biſected by the ſtraight line FMN, the ſides EF, FG together have to EG the third ſide the ratio of FE to EM, that is, of H to K. In like manner, the ſides EF, FO together have to EQ the ratio which H has to K.

f. 18. 1.

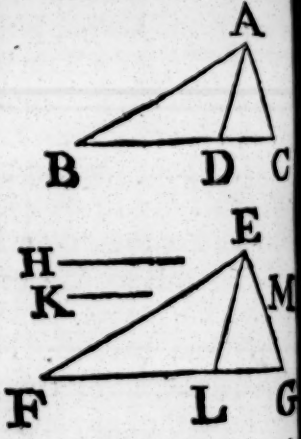
PROP.

46. PROP. XLIX.

IF a triangle has one angle given, and if the ſides abo another angle, both together, have a given ratio the third ſide; the triangle is given in ſpecies.

Let the triangle ABC have one angle ABC given, and the two ſides BA, AC about another angle BAC have a giv ratio to BC; the triangle ABC is given in ſpecies.

Suppoſe the angle BAC to be biſected by the ſtraight li AD; BA and AC together are to BC, as AB to BD, as w ſhewn in the preceeding Propoſition. But the ratio of B and AC together to BC is given, therefore alſo the ratio AB to BD is given. And the angle ABD is given, wher a.44.Dat. fore [a] the triangle ABD is given in ſpecies; and conſequen ly the angle BAD, and its double the angle BAC are given; and the angle ABC is given. Therefore the b.43.Dat. triangle ABC is given in ſpecies[b].



A triangle which ſhall have the things mentioned in the Propoſition to be given, may be thus found. Let EFG be the given angle, and the ratio of H to K the given ratio; and by Prop. 44. find the triangle EFL, which has the angle EFG for one of its angles, and the ratio of the ſides EF, FL about this angle the ſame with the ratio of H to K; and make the angle LEM equal to th angle FEL. And becauſe the ratio of H to K is the ratio whic two ſides of a triangle have to the third, H muſt be greater tha K; and becauſe EF is to FL, as H to K, therefore EF is greate than FL, and the angle FEL, that is, LEM, is therefore leſs tha the angle ELF. Wherefore the angles LFE, FEM are leſs tha two right angles, as was ſhewn in the foregoing Propoſition and the ſtraight lines FL, EM muſt meet if produced; le them meet in G, EFG is the triangle which was to be found for EFG is one of its angles, and becauſe the angle FEG i biſected by EL, the two ſides FE, EG together have to th third ſide FG the ratio of EF to FL, that is, the given ratio o H to K.

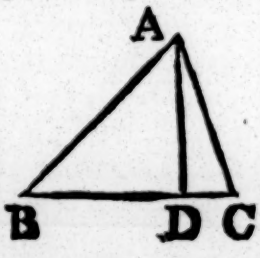
PROP

PROP. L. 76.

IF from the vertex of a triangle given in ſpecies, a ſtraight line be drawn to the baſe in a given angle; ſhall have a given ratio to the baſe.

From the vertex A of the triangle ABC which is given in ecies, let AD be drawn to the baſe BC in a given angle ADB; e ratio of AD to BC is given.

Becauſe the triangle ABC is given in ecies, the angle ABD is given, and the gle ADB is given; therefore the triangle BD is given [a] in ſpecies; wherefore the tio of AD to AB is given. And the ratio AB to BC is given; and therefore [b] the tio of AD to BC is given.



a 43. Dat.

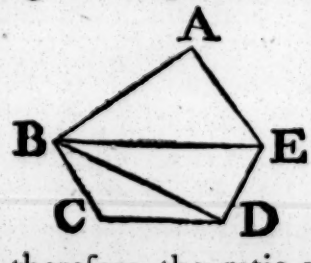
b 9. Dat.

PROP. LI. 47.

RECTILINEAL figures given in ſpecies, are divided into triangles which are given in ſpecies.

Let the rectilineal figure ABCDE be given in ſpecies; BCDE may be divided into triangles given in ſpecies.

Join BE, BD; and becauſe ABCDE is given in ſpecies, the gle BAE is given [a], and the ratio of A to BE is given [a]; wherefore the iangle BAE is given in ſpecies [b], and e angle AEB is therefore given [a]. But e whole angle AED is given, and erefore the remaining angle BED is iven, and the ratio of AE to EB is iven, as alſo the ratio of AE to ED; therefore the ratio of E to ED is given [c]. And the angle BED is given, wherefore e triangle BED is given [b] in ſpecies. In the ſame manner, the iangle BDC is given in ſpecies: Therefore rectilineal figures hich are given in ſpecies are divided into triangles given in ecies.



a 3. Def.

b 44. Dat.

c 9. Dat.

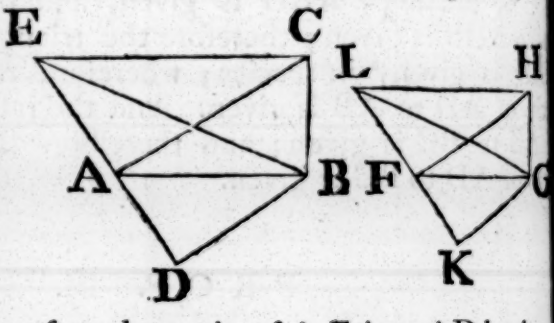
PROP.

48. PROP. LII.

IF two triangles given in ſpecies be deſcribed upon th ſame ſtraight line; they ſhall have a given ratio t one another.

Let the triangles ABC, ABD given in ſpecies be deſcribe upon the ſame ſtraight line AB; the ratio of the triangle AB to the triangle ABD is given.

Thro' the point C, draw CE parallel to AB, and let it me DA produced in E, and join BE. Becauſe the triangle AB is given in ſpecies, the angle BAC, that is, the angle ACE, given; and becauſe the triangle ABD is given in ſpecies, th angle DAB, that is, the angle AEC, is given. Therefore the triangle ACE is given in ſpecies; wherefore the ratio of EA to AC



a 3. Def. is given[a], and the ratio of CA to AB is given, as alſo the ratio of BA to AD; therefore the ratio of[b] EA to AD is given

b 9. Dat.

c 37. 1. and the triangle ACB is equal[c] to the triangle AEB, and

d 1. 6. the triangle AEB, or ACB, is to the triangle ADB, ſo is[d] th ſtraight line EA to AD. But the ratio of EA to AD is given therefore the ratio of the triangle ACB to the triangle ADB given.

PROBLEM.

To find the ratio of two triangles ABC, ABD given in ſpecies, and which are deſcribed upon the ſame ſtraight line AB.

Take a ſtraight line FG given in poſition and magnitude and becauſe the angles of the triangles ABC, ABD are given at the points F, G of the ſtraight line FG, make the angle GFH, GFK[e] equal to the angles BAC, BAD; and the angle

e 23. 1. FGH, FGK equal to the angles ABC, ABD, each to each Therefore the triangles ABC, ABD are equiangular to the t angles FGH, FGK, each to each. Thro' the point H, draw H parallel to FG meeting KF produced in L. And becauſe th angles BAC, BAD are equal to the angles GFH, GFK, eac to each; therefore the angles ACE, AEC are equal to FH FLH, each to each, and the triangle AEC equiangular to t triangle FLH. Therefore as EA to AC, ſo is LF to FH; a

CA to AB, ſo HF to FG; and as BA to AD, ſo is GF FK; wherefore, ex æquali, as EA to AD, ſo is LF to K. But, as was ſhewn, the triangle ABC is to the triangle BD, as the ſtraight line EA to AD, that is, as LF to FK. he ratio therefore of LF to FK has been found, which is the me with the ratio of the triangle ABC to the triangle ABD.

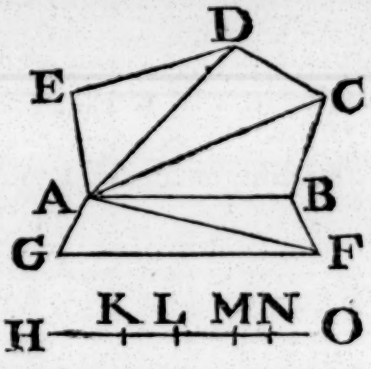
PROP. LIII. 49.

F two rectilineal figures given in ſpecies be deſcribed upon the ſame ſtraight line; they ſhall have a given tio to one another. See N.

Let any two rectilineal figures ABCDE, ABFG which are ven in ſpecies, be deſcribed upon the ſame ſtraight line AB; e ratio of them to one another is given.

Join AC, AD, AF; each of the triangles AED, ADC, CB, AGF, ABF is given[a] in ſpecies. And becauſe the tri- a 51. Dat.

gles ADE, ADC given in ſpecies e deſcribed upon the ſame ſtraight ne AD, the ratio of EAD to AC is given[b]; and, by compoſition, e ratio of EACD to DAC is ven[c]. And the ratio of DAC to AB is given[b], becauſe they are ſcribed upon the ſame ſtraight ne AC; therefore the ratio of ACD to ACB is given[d]; and, by mpoſition, the ratio of ABCDE ABC is given. In the ſame manner, the ratio of ABFG to BF is given. But the ratio of the triangle ABC to the triangle BF is given; wherefore[b], becauſe the ratio of ABCDE to ABC given, as alſo the ratio of ABC to ABF, and the ratio of ABF ABFG; the ratio of the rectilineal ABCDE to the rectilineal BFG is given[d].



b 52. Dat.

c 7. Dat.

d 9. Dat.

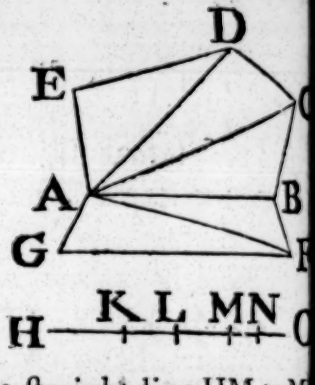
PROBLEM.

To find the ratio of two rectilineal figures given in ſpecies, d deſcribed upon the ſame ſtraight line.

Let ABCDE, ABFG be two rectilineal figures given in ecies, and deſcribed upon the ſame ſtraight line AB, and n AC, AD, AF. Take a ſtraight line HK given in poſition d magnitude, and by the 52. Dat. find the ratio of the tri- gle ADE to the triangle ADC, and make the ratio of HK

to

to KL the ſame with it. Find alſo the ratio of the triang ACD to the triangle ACB. And make the ratio of KL LM the ſame. Alſo, find the ratio of the triangle ABC to t triangle ABF, and make the ratio of LM to MN the ſam And laſtly, find the ratio of the triangle AFB to the triang AFG, and make the ratio of MN to NO the ſame. Then the ratio of ABCDE to ABFG is the ſame with the ratio of HM to MO.



Becauſe the triangle EAD is to the triangle DAC, as the ſtraight line HK to KL; and as the triangle DAC to CAB, ſo is the ſtraight line KL to LM; therefore, by uſing compoſition as often as the number of triangles requires, the rectilineal ABCDE is to the triangle ABC, as the ſtraight line HM to M In like manner, becauſe the triangle GAF is to FAB, as ON NM, by compoſition, the rectilineal ABFG is to the triangle AB as MO to MN; and, by inverſion, as ABF to ABFG, ſo is N to MO. And the triangle ABC is to ABF, as LM to M Wherefore, becauſe as ABCDE to ABC, ſo is HM to ML; a as ABC to ABF, ſo is LM to MN; and as ABF to ABFG, ſo MN to MO; ex æquali, as the rectilineal ABCDE to ABFG, is the ſtraight line HM to MO.

50. PROP. LIV.

IF two ſtraight lines have a given ratio to one anothe the ſimilar rectilineal figures deſcribed upon them milarly, ſhall have a given ratio to one another.

Let the ſtraight lines AB, CD have a given ratio to another, and let the ſimilar and ſimilarly placed rectilineal gures E, F be deſcribed upon them; the ratio of E to F given.

To AB, CD, let G be a third proportional; therefore as AB to CD, ſo is CD to G. And the ratio of AB to CD is given, wherefore the ratio of CD to G is given; and conſequently the ratio of AB to G is alſo given [a]. But as AB to G, ſo is the figure E to the figure [b] F. Therefore the ratio of E to given.

a 9. Dat.
b 2. Cor. 20. 6.

PR

PROBLEM.

To find the ratio of two ſimilar rectilineal figures E, F ſimilarly deſcribed upon ſtraight lines AB, CD which have given ratio to one another: Let G be a third proportional to AB, CD.

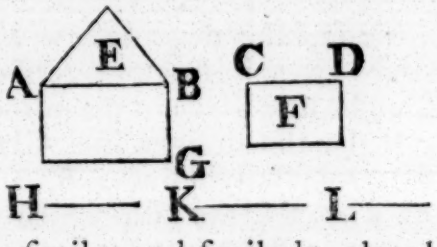
Take a ſtraight line H given in magnitude; and becauſe the ratio of AB to CD is given, make the ratio of H to K the ſame with it; and becauſe H is given, K is given. As H is to K, ſo make K to L; then the ratio of E to F is the ſame with the ratio of H to L; for AB is to CD, as H to K, wherefore CD is to G, as K to L; and, ex equali, as AB to G, ſo is H to L: But the figure E is to [b] the figure F, as AB to G, that is, as H to L. [b] 2 Cor. 20. 6.

PROP. LV.

IF two ſtraight lines have a given ratio to one another; 51. the rectilineal figures given in ſpecies deſcribed upon them, ſhall have to one another a given ratio.

Let AB, CD be two ſtraight lines which have a given ratio to one another; the rectilineal figures E, F given in ſpecies and deſcribed upon them, have a given ratio to one another.

Upon the ſtraight line AB, deſcribe the figure AG ſimilar and ſimilarly placed to the figure F; and becauſe F is given in ſpecies, AG is alſo given in ſpecies: Therefore, ſince the figures E, AG which are given in ſpecies, are deſcribed upon the ſame ſtraight line AB, the ratio of E to AG is given [a]. and becauſe the ratio of AB to CD is given, and upon them are deſcribed the ſimilar and ſimilarly placed rectilineal figures AG, F, the ratio of AG to F is given [b]; and the ratio of AG to E is given; therefore the ratio of E to F is given [c].



[a] 53. Dat.

[b] 54. Dat.

[c] 9. Dat.

PROBLEM.

To find the ratio of two rectilineal figures E, F given in ſpecies, and deſcribed upon the ſtraight lines AB, CD which have a given ratio to one another.

Take a ſtraight line H given in magnitude; and becauſe the rectilineal figures E, AG given in ſpecies are deſcribed upon the ſame ſtraight line AB, find their ratio by the 53. Dat. and make the ratio of H to K the ſame; K is therefore given; And becauſe the ſimilar rectilineal figures AG, F are

described upon the ſtraight lines AB, CD which have a give ratio, find their ratio by the 54. Dat. and make the ratio of to L the ſame: The figure E has to F the ſame ratio which H h to L: For, by the conſtruction, as E is to AG, ſo is H to K and as AG to F, ſo is K to L; therefore, ex equali, as E to F ſo is H to L.

52. PROP. LVI.

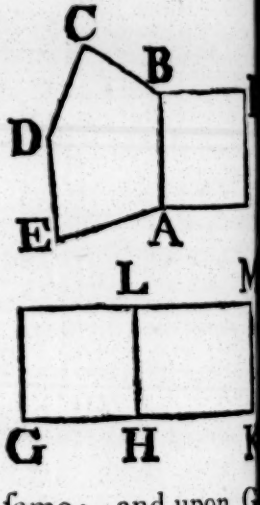
IF a rectilineal figure given in ſpecies be deſcribed u on a ſtraight line given in magnitude; the figure is g ven in magnitude.

Let the rectilineal figure ABCDE given in ſpecies be deſcrib upon the ſtraight line AB given in magnitude; the figu ABCDE is given in magnitude.

Upon AB let the ſquare AF be deſcribed; therefo AF is given in ſpecies and magnitude, and becauſe t rectilineal figures ABCDE, AF given in ſpecies are deſcribed upon the ſame ſtraight line AB, the ratio of ABCDE to AF is given [a]: But the ſquare AF is given in magnitude, therefore [b] alſo the figure ABCDE is given in magnitude.

[a]. 53 Dat.

[b]. 2. Dat



PROB.

To find the magnitude of a rectilineal figure given in ſpecies deſcribed upon a ſtraight line given in magnitude.

Take the ſtraight line GH equal to the given ſtraight line AB, and by the 53. Dat. find the ratio which the ſquare AF upon AB has to the figure ABCDE; and make the ratio of GH to HK the ſame; and upon G deſcribe the ſquare GL, and complete the parallelogra LHKM; the figure ABCDE is equal to LHKM: Becau AF is to ABCDE, as the ſtraight line GH to HK, that is, the figure GL to HM; and AF is equal to GL; therefo ABCDE is equal to HM [c].

[c] 14. 5.

53. PROP. LVII.

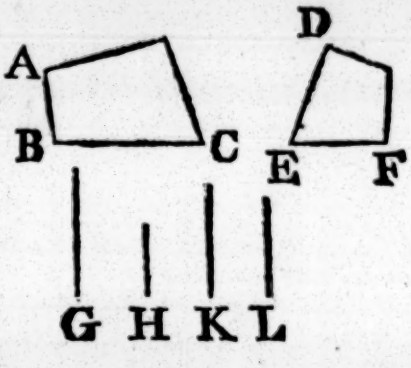
IF two rectilineal figures are given in ſpecies, and i ſide of one of them has a given ratio to a ſide of t other; the ratios of the remaining ſides to the remaini ſides ſhall be given.

Let AC, DF be two rectilineal figures given in ſpecies, and let the ratio of the ſide AB to the ſide DE be given; the ratios of the remaining ſides to the remaining ſides are alſo given.

Becauſe the ratio of AB to DE is given, as alſo [a] the ratios of AB to BC, and of DE to EF; the ratio of BC to EF is given[b]. In the ſame manner, the ratios of the other ſides to the other ſides are given.

a 3. Def.

b 10. Dat.



The ratio which BC has to EF may be found thus; take a ſtraight line G given in magnitude, and becauſe the ratio of BC to BA is given, make the ratio of G to H the ſame; and becauſe the ratio of AB to DE is given, make the ratio of H to K the ſame; and make the ratio of K to L the ſame with the given ratio of DE to EF. Since therefore as BC to BA, ſo is G to H; and as BA to DE, ſo is H to K; and as DE to EF, ſo is K to L; ex æquali, BC is to EF, as G to L; therefore the ratio of G to L has been found, which is the ſame with the ratio of BC to EF.

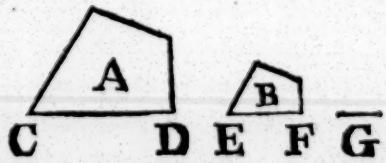
PROP. LVIII. G.

IF two ſimilar rectilineal figures have a given ratio to one another; their homologous ſides have alſo a given ratio to one another. See N.

Let the two ſimilar rectilineal figures A, B have a given ratio to one another; their homologous ſides have alſo a given ratio.

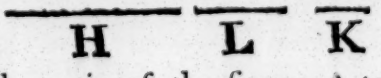
Let the ſide CD be homologous to EF, and to CD, EF let the ſtraight line G be a third proportional. As therefore [a] CD to G, ſo is the figure A to B; and the ratio of A to B is given, therefore the ratio of CD to G is given; and CD, EF, G are proportionals; wherefore [b] the ratio of CD to EF is given.

a 2. Cor. 20. 6.



b 13. Dat.

The ratio of CD to EF may be found thus; take a ſtraight line H given in magnitude; and becauſe the ratio of the figure A to B is given, make the ratio of H to K the ſame with it: And, as the 13. Dat. directs to be done, find a mean proportional L between



between H and K; the ratio of CD to EF is the ſame with that of H to L. Let G be a third proportional to CD, EF; therefore as CD to G, ſo is (A to B, and ſo is) H to K; and as CD to EF, ſo is H to L, as is ſhewn in the 13. Dat.

PROP. LIX.

54. See N.

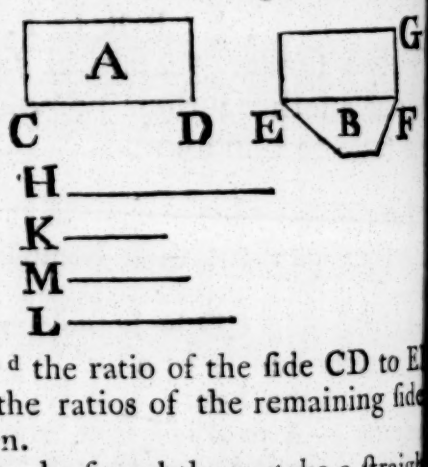
IF two rectilineal figures given in ſpecies have a given ratio to one another; their ſides ſhall likewiſe have given ratios to one another.

Let the two rectilineal figures A, B given in ſpecies, have a given ratio to one another; their ſides ſhall alſo have given ratios to one another.

If the figure A be ſimilar to B, their homologous ſides ſhall have a given ratio to one another, by the preceeding Propoſition; and becauſe the figures are given in ſpecies, the ſides of each of them bave given ratios [a] to one another; therefore each ſide of one of them has [b] to each ſide of the other a given ratio.

a 3. Def.
b 9. Dat.

But if the figure A be not ſimilar to B, let CD, EF be any two of their ſides; and upon EF conceive the figure EG to be deſcribed ſimilar and ſimilarly placed to the figure A, ſo that CD, EF be homologous ſides; therefore EG is given in ſpecies; and the figure B is given in ſpecies; wherefore [c] the ratio of B to EG is given; and the ratio of A to B is given, therefore [b] the ratio of the figure A to EG is given; and A is ſimilar to EG; therefore [d] the ratio of the ſide CD to E is given; and conſequently [b] the ratios of the remaining ſide to the remaining ſides are given.



c 53. Dat.
d 58. Dat.

The ratio of CD to EF may be found thus; take a ſtraight line H given in magnitude, and becauſe the ratio of the figure A to B is given, make the ratio of H to K the ſame with it. And by the 53. Dat. find the ratio of the figure B to EG, and make the ratio of K to L the ſame: Between H and L find a mean proportional M; the ratio of CD to EF is the ſame with the ratio of H to M; becauſe the figure A is to B as H to K; and as B to EG, ſo is K to L; ex æquali, as

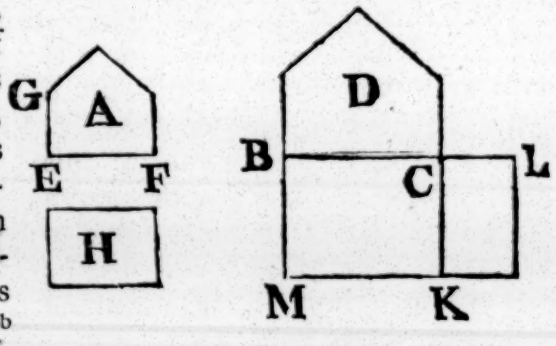
to EG, ſo is H to L; And the figures A, EG are ſimilar, and M is a mean proportional between H and L; therefore, as was ſhewn in the preceeding Propoſition, CD is to EF, as H to M.

PROP. LX. 55.

IF a rectilineal figure be given in ſpecies and magnitude, the ſides of it ſhall be given in magnitude.

Let the rectilineal figure A be given in ſpecies and magnitude; its ſides are given in magnitude.

Take a ſtraight line BC given in poſition and magnitude; and upon BC deſcribe [a] the figure D ſimilar, and ſimilarly placed, to the figure A, and let EF be the ſide of the figure A homologous to BC the ſide of D; therefore the figure D is given in ſpecies. And becauſe upon the given ſtraight line BC the figure D given in ſpecies is deſcribed, D is given [b] in magnitude, and the figure A is given in magnitude, therefore the ratio of A to D is given: And the figure A is ſimilar to D; therefore the ratio of the ſide EF to the homologous ſide BC is given [c]; and BC is given, wherefore [d] EF is given: And the ratio of EF to EG is given [e], therefore EG is given. And, in the ſame manner, each of the other ſides of the figure A can be ſhewn to be given.



a 18. 6.

b 56. Dat.

c 58. Dat.
d 2. Dat.
e 3. Def.

PROBLEM.

To deſcribe a rectilineal figure A ſimilar to a given figure D, and equal to another given figure H. It is Prop. 25. B. 6. Elem.

Becauſe each of the figures, D H is given, their ratio is given, which may be found by making [f] upon the given ſtraight line BC the parallelogram BK equal to D, and upon its ſide CK making [f] the parallelogram KL equal to H in the angle KCL equal to the angle MBC; therefore the ratio of D to H, that is, of BK to KL, is the ſame with the ratio of BC to CL: And becauſe the figures D, A are ſimilar, and that the ratio of D to A, or H, is the ſame with the ratio of BC to CL; by the 58. Dat. the ratio of the homologous ſides BC, EF is the ſame with the ratio of BC to the mean proportional between BC and CL. Find EF the mean proportional; then EF is the

f Cor. 45. 1.

ſide of the figure to be deſcribed, homologous to BC the ſide of D, and the figure itſelf can be deſcribed by the 18th Prop. B. 6. which, by the conſtruction, is ſimilar to D; and becauſe
g 2. Cor. 20. 6. D is to A, as [g] BC to CL, that is as the figure BK to KL; and that D is equal to BK, therefore A
h 14. 5. [h] is equal to KL, that is, to H.

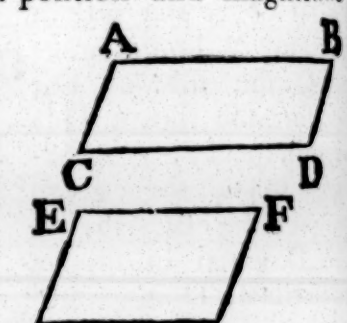
57. See N.

PROP LXI.

IF a parallelogram given in magnitude has one of its ſides and one of its angles given in magnitude; the other ſide alſo is given.

Let the parallelogram ABCD given in magnitude, have the ſide AB and the angle BAC given in magnitude; the other ſide AC is given.

Take a ſtraight line EF given in poſition and magnitude; and becauſe the parallelogram AD is given in magnitude, a rectilineal



a 1. Def. figure equal to it can be found [a]. And a parallelogram equal to this
b Cor. 45. 1. figure can be applied [b] to the given ſtraight line EF in an angle equal to the given angle BAC. Let this be the parallelogram EFGH having the angle FEG equal to the angle BAC. And becauſe the parallelograms AD, EH are equal, and have the angles at A and E equal; the ſides about them are reciprocally proportional
c 14. 6. d 12. 6. [c]; therefore as AB to EF, ſo is EG to AC; and AB, EF, EG are given, therefore alſo AC is given [d]. Whence the way of finding AC is manifeſt.

H. See N.

PROP. LXII.

IF a parallelogram has a given angle, the rectangle contained by the ſides about that angle has a given ratio to the parallelogram.

Let the parallelogram ABCD have the given angle ABC; the rectangle AB, BC has a given ratio to the parallelogram AC.

From the point A draw AE perpendicular to BC; becauſe the angle ABC is given, as alſo the angle AEB; the triangle
a 43. Dat. ABE is given [a] in ſpecies; therefore the ratio of BA to AE is given. But as BA
b. 1. 6. to AE, ſo is [b] the rectangle AB, BC to the rectangle AE, BC; therefore the ratio of

the

the rectangle AB, BC to AE, BC, that is[c], to the parallelogram AC, is given. c. 35. 1.

And it is evident how the ratio of the rectangle to the parallelogram may be found, by making the angle FGH equal to the given angle ABC, and drawing, from any point F in one of its ſides, FK perpendicular to the other GH; for GF is to FK, as BA to AE, that is, as the rectangle AB, BC, to the parallelogram AC.

Cor. And if a triangle ABC has a given angle ABC, the rectangle AB, BC contained by the ſides about that angle, ſhall have a given ratio to the triangle ABC. 66.

Complete the parallelogram ABCD; therefore, by this Propoſition, the rectangle AB, BC has a given ratio to the parallelogram AC; and AC has a given ratio to its half the triangle[d] ABC; therefore the rectangle AB, BC has a given[e] ratio to the triangle ABC. d. 14. 1. e. 9. Dat.

And the ratio of the rectangle to the triangle is found thus; make the triangle FGK as was ſhewn in the Propoſition; the ratio of GF to the half of the perpendicular FK is the ſame with the ratio of the rectangle AB, BC to the triangle ABC. Becauſe, as was ſhewn, GF is to FK, as AB, BC to the parallelogram AC; and FK is to its half, as AC is to its half, which is the triangle ABC; therefore, ex aequali, GF is to the half of FK, as AB, BC rectangle is to the triangle ABC.

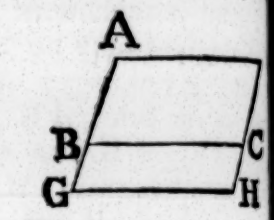
PROP. LXIII. 56.

IF two parallelograms be equiangular, as a ſide of the firſt to a ſide of the ſecond, ſo is the other ſide of the ſecond to the ſtraight line to which the other ſide of the firſt has the ſame ratio which the firſt parallelogram has to the ſecond. And conſequently, if the ratio of the firſt parallelogram to the ſecond be given, the ratio of the other ſide of the firſt to that ſtraight line is given; and if the ratio of the other ſide of the firſt to that ſtraight line be given, the ratio of the firſt parallelogram to the ſecond is given.

Let AC, DF be two equiangular parallelograms, as BC a ſide of the firſt is to EF a ſide of the ſecond, ſo is DE the other ſide of the ſecond to the ſtraight line to which AB the o-

ther ſide of the firſt has the ſame ratio which AC has to DF.

Produce the ſtraight line AB, and make as BC to EF, ſo DE to BG, and complete the parallelogram BGHC; therefore, becauſe BC, or GH, is to EF, as DE to BG, the ſides about the equal angles BGH, DEF are
a. 14. 6. reciprocally proportional; wherefore [a] the parallelogram BH is equal to DF; and AB is to BG, as the parallelogram AC is to BH, that is, to DF; as therefore BC is to EF, ſo is DE to BG which is the ſtraight line to which AB has the ſame ratio that AC has to DF.



And if the ratio of the parallelogram AC to DF be given, then the ratio of the ſtraight line AB to BG is given; and if the ratio of AB to the ſtraight line BG be given, the ratio of the parallelogram AC to DF is given.

74. 73.

PROP. XLIV.

See N.

IF two parallelograms have unequal, but given, angles, and if as a ſide of the firſt to a ſide of the ſecond, ſo the other ſide of the ſecond be made to a certain ſtraight line; if the ratio of the firſt parallelogram to the ſecond be given, the ratio of the other ſide of the firſt to that ſtraight line ſhall be given. And if the ratio of the other ſide of the firſt to that ſtraight line be given, the ratio of the firſt parallelogram to the ſecond ſhall be given.

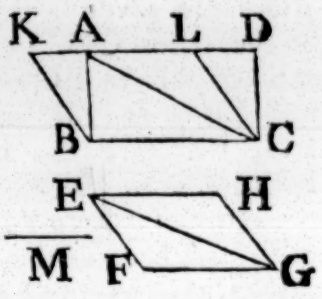
Let ABCD, EFGH be two parallelograms which have the unequal, but given, angles ABC, EFG; and as BC to FG, ſo make EF to the ſtraight line M. If the ratio of the parallelogram AC to EG be given, the ratio of AB to M is given.

At the point B of the ſtraight line BC make the angle CBK equal to the angle EFG, and complete the parallelogram KBCL. And becauſe the ratio of AC to EG is given, and that
a. 35. 1. AC is equal [a] to the parallelogram KC, therefore the ratio of KC to EG is given; and KC, EG are equiangular; there-
b. 63. Dat. fore as BC to FG, ſo is [b] EF to the ſtraight line to which KB has a given ratio, viz. the ſame which the parallelogram KC has to EG: But as BC to FG, ſo is EF to the ſtraight line M; therefore KB has a given ratio to M; and the ratio

of

AB to BK is given, becauſe the triangle ABK is given in ecies[c]; therefore the ratio of AB to M is given[d]. c.43.Dat. d. 9. Dat.

And if the ratio of AB to M be given, the ratio of the pa- llelogram AC to EG is given; for ſince the ratio of KB to A is given, as alſo the ratio of AB M, the ratio of KB to M is given[d]; nd becauſe the parallelograms KC, EG e equiangular, as BC to FG, ſo is EF to the ſtraight line to which KB s the ſame ratio which the parallelo- am KC has to EG; but as BC to FG, is EF to M; therefore KB is to M, the parallelogram KC is to EG; and e ratio of KB to M is given, therefore the ratio of the paral- ogram KC, that is, of AC to EG, is given.



b.63.Dat.

Cor. And if two triangles ABC, EFG have two equal gles, or two unequal, but given, angles ABC, EFG, and as BC a ſide of the firſt to FG a ſide of the ſecond, ſo the her ſide of the ſecond EF be made to a ſtraight line M; if e ratio of the triangles be given, the ratio of the other ſide the firſt to the ſtraight line M is given. 75.

Complete the parallelograms ABCD, EFGH; and becauſe e ratio of the triangle ABC to the triangle EFG is given, e ratio of the parallelogram AC to EG is given[e], becauſe e. 15. 5. e parallelograms are double[f] of the triangles; and becauſe f. 41. 1. C is to FG, as EF to M, the ratio of AB to M is given by e 63. Dat. if the angles ABC, EFG are equal; but if they unequal, but given angles, the ratio of AB to M is given this Propoſition.

And if the ratio of AB to M be given, the ratio of the pa- llelogram AC to EG is given by the ſame Propoſitions; and refore the ratio of the triangle ABC to EFG is given.

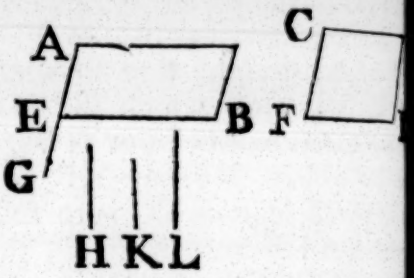
PROP. LXV. 68.

F two equiangular parallelograms have a given ratio to one another, and if one ſide has to one ſide a gi- n ratio; the other ſide ſhall alſo have to the other ſide iven ratio.

Let the two equiangular parallelograms AB, CD have a en ratio to one another, and let the ſide EB have a given o to the ſide FD; the other ſide AE has alſo a given ratio he other ſide CF.

Becauſe

Becauſe the two equiangular parallelograms AB, CD h
a given ratio to one another; as EB a ſide of the firſt is to
a. 63. Dat. a ſide of the ſecond, ſo is [a] FC the other ſide of the ſecond
the ſtraight line to which AE the other ſide of the firſt
the ſame given ratio which the firſt parallelogram AB
to the other CD. Let this ſtraight line be EG; therefore
ratio of AE to EG is given;
and EB is to FD, as FC to
EG, therefore the ratio of
FC to EG is given, becauſe
the ratio of EB to FD is gi-
ven; and becauſe the ratio of
AE to EG, as alſo the ratio
of FC to EG is given; the ra-
b. 9. Dat. tio of AE to CF is given [b].



The ratio of AE to CF may be found thus; take a ſtrai
line H given in magnitude; and becauſe the ratio of the
rallelogram AB to CD is given, make the ratio of H to K
ſame with it. And becauſe the ratio of FD to EB is given, m
the ratio of K to L the ſame: The ratio of AE to CF is
ſame with the ratio of H to L. Make as EB to FD, ſo FC
EG, therefore, by inverſion, as FD to EB, ſo is EG to F
and as AE to EG, ſo is [a] (the parallelogram AB to CD, a
ſo is) H to K; but as EG to FC, ſo is (FD to EB, and ſo
K to L; therefore, ex aequali, as AE to FC, ſo is H to L.

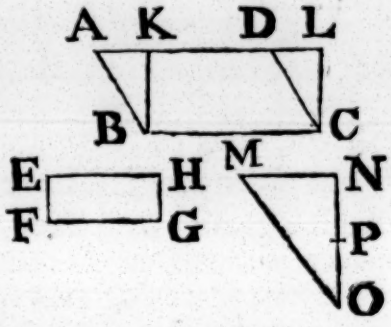
69. PROP. LXVI.

IF two parallelograms have unequal, but given, angl
and a given ratio to one another; if one ſide has
one ſide a given ratio, the other ſide has alſo a given
tio to the other ſide.

Let the two parallelograms ABCD, EFGH which have
given unequal angles ABC, EFG, have a given ratio to
another, and let the ratio of BC to FG be given; the ra
alſo of AB to EF is given.

At the point B of the ſtraight line BC make the angle C
equal to the given angle EFG, and complete the paralle
gram BKLC; and becauſe each of the angles BAK, AKB
a. 43. Dat. given, the triangle ABK is given [a] in ſpecies; therefore
ratio of AB to BK is given; and becauſe, by the hypothe

ratio of the parallelogram AC to EG is given, and that C is equal [b] to BL; therefore the ratio of BL to EG is gi- [b. 35. 1.
n: And becauſe BL is equiangular to EG, and by the hypo-
ſis, the ratio of BC to FG is given; therefore [c] the ratio [c 65 Dat.
KB to EF is given, and the ra-
of KB to BA is given; the
io therefore [d] of AB to EF is [d. 9. Dat.
en.

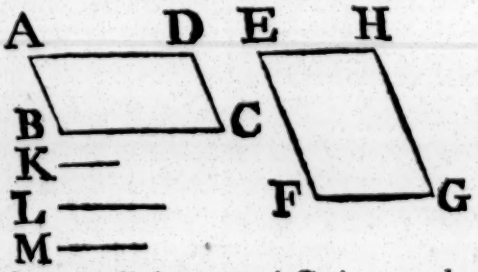


The ratio of AB to EF may be
nd thus; take the ſtraight line
N given in poſition and magni-
e; and make the angle NMO
al to the given angle BAK,
the angle MNO equal to the
en angle EFG or AKB: And becauſe the parallelogram BL
quiangular to EG, and has a given ratio to it, and that
ratio of BC to FG is given; find by the 65. Dat. the ratio
KB to EF; and make the ratio of NO to OP the ſame with
Then the ratio of AB to EF is the ſame with the ratio of
O to OP: For, ſince the triangle ABK is equiangular to
ON, as AB to BK, ſo is MO to ON; and as KB to EF, ſo
NO to OP; therefore, ex aequali, as AB to EF, ſo is MO to
.

PROP. LXVII. 70.

the ſides of two equiangular parallelograms have gi- See N.
ven ratios to one another; the parallelograms ſhall
e a given ratio to one another.

Let ABCD, EFGH be two equiangular parallelograms, and
he ratio of AB to EF, as alſo the ratio of BC to FG, be given;
ratio of the parallelogram AC to EG is given.
Take a ſtraight line K given in magnitude, and becauſe the
o of AB to EF is given,
e the ratio of K to L the
e with it; therefore L is
n [a]: And becauſe the ratio
C to FG is given, make
ratio of L to M the ſame;
refore M is given [a]; and
s given, wherefore [b] the
o of K to M is given: But the parallelogram AC is to the
llelogram EG, as the ſtraight line K to the ſtraight line M,



[a. 2. Dat.
[b. 1. Dat.

as

as is demonſtrated in the 23. Prop. of B. 6. Elem. therefore ratio of AC to EG is given.

From this it is plain how the ratio of two equiangular rallelograms may be found when the ratios of their ſides given.

70. PROP. LXVIII.

See N. IF the ſides of two parallelograms which have unequ but given, angles, have given ratios to one another; parallelograms ſhall have a given ratio to one another.

Let two parallelograms ABCD, EFGH which have the ven unequal angles ABC, EFG have the ratios of their ſi viz. of AB to EF, and of BC to FG, given; the ratio of the rallelogram AC to EG is given

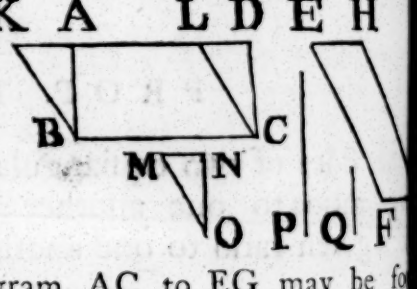
At the point B of the ſtraight line BC make the angle C equal to the given angle EFG, and complete the parall gram KBCL: And becauſe each of the angles BAK, B

a. 43. Dat. is given, the triangle ABK is given [a] in ſpecies: Therefore ratio of AB to BK is given; and the ratio of AB to E

b. 9. Dat. given, wherefore [b] the ratio of BK to EF is given: And ratio of BC to FG is given; and the angle KBC is equal to the angle EFG; there-

c. 67. Dat. fore [c] the ratio of the parallelogram KC to EG is gi-

d. 35. 1. ven: But KC is equal [d] to AC; therefore the ratio of AC to EG is given.



The ratio of the parallelogram AC to EG may be fo thus; take the ſtraight line MN given in poſition and ma tude, and make the angle MNO equal to the given a KAB, and the angle NMO equal to the given angle AK FEH: And becauſe the ratio of AB to EF is given, make ratio of NO to P the ſame; alſo make the ratio of P to Q ſame with the given ratio of BC to FG, the parallelogram A to EG, as MO to Q.

Becauſe the angle ABK is equal to the angle MNO, the angle AKB equal to the angle NMO; the triangle A is equiangular to NMO: Therefore as KB to BA, ſo is M ON; and as BA to EF, ſo is NO to P; wherefore, ex quali, as KB to EF, ſo is MO to P: And BC is to FG,

Q, and the parallelograms KC, EG are equiangular; there-
e, as was ſhewn in Prop. 67. the parallelogram KC, that is,
C, is to EG, as MO to Q.

Cor. 1. If two triangles ABC, DEF have two equal angles, 71.
two unequal, but given angles ABC, DEF, and if the ra-
s of the ſides about theſe angles,
. the ratios of AB to DE, and
BC to EF be given; the triangles
ll have a given ratio to one ano-
r.

Complete the parallelograms BG,
H; the ratio of BG to EH is given [a]; and therefore the tri- a 67. or
gles which are the halves [b] of them have a given [c] ratio to one 68. Dat.
ther. b 34. 1.
c 15. 5.

Cor. 2. If the baſes BC, EF of two triangles ABC, DEF have 72.
iven ratio to one another, and if alſo the ſtraight lines AG,
H which are drawn to the baſes from the oppoſite angles,
her in equal angles, or unequal, but given, angles AGC,
HF have a given ratio to one
ther; the triangles ſhall have
iven ratio to one another.
Draw BK, EL parallel to AG,
H, and complete the paralle-
rams KC, LF. And becauſe the angles AGC, DHF, or
ir equals, the angles KBC, LEF are either equal, or unequal,
given; and that the ratio of AG to DH; that is, of KB to
, is given, as alſo the ratio of BC to EF; therefore [a] the ratio a 67. or
he parallelogram KC to LF is given; wherefore alſo the ra- 68. Dat.
of the triangle ABC to DEF is given [b]. b. { 41. 1. / 15. 5.

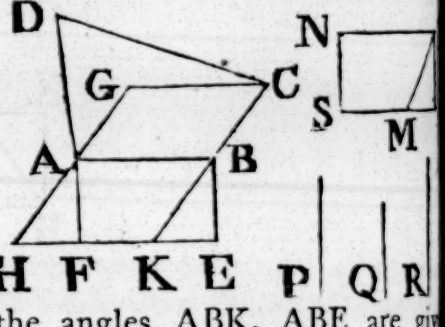
PROP. LXIX. 61.

a parallelogram which has a given angle be applied
to one ſide of a rectilineal figure given in ſpecies; if
figure have a given ratio to the parallelogram, the pa-
elogram is given in ſpecies.

Let ABCD be a rectilineal figure given in ſpecies, and to one
of it AB, let the parallelogram ABEF having the given
gle ABE be applied; if the figure ABCD has a given ratio
he parallelogram BF, the parallelogram BF is given in ſpe-
s.

Thro' the point A draw AG parallel to BC, and through the
nt C draw CG parallel to AB, and produce GA, CB to
the

a 3. Def. the points H, K; becauſe the angle ABC is given [a], and ratio of AB to BC is given, the figure ABCD being given ſpecies; therefore, the parallelogram BG is given [a] in ſpec And becauſe upon the ſame ſtraight line AB the two rectilin figures BD, BG given in ſpecies are deſcribed, the ratio
b 53. Dat. BD to BG is given [b]; and, by hypotheſis, the ratio
c 9. Dat. BD to the parallelogram BF is given; wherefore [c] the ratio
d 35. 1. BF, that is [d], of the parallelogram BH, to BG is given,
e 1. 6. therefore [e] the ratio of the ſtraight line KB to BC is giv and the ratio of BC to BA is given, wherefore the ratio KB to BA is given[c]: And becauſe the angle ABC is given, adjacent angle ABK is given; and the angle ABE is giv therefore the remaining angle KBE is given. The angle E is alſo given, becauſe it is equal to the angle ABK; theref the triangle BKE is given in ſpecies, and conſequently the tio of EB to BK is given; and the ratio of KB to BA is giv wherefore [c] the ratio of EB to BA is given; and the angle ABE is given, therefore the parallelogram BF is given [a] in ſpecies.



A parallelogram ſimilar to BF may be found thus; take a ſtraight line LM given in poſition and magnitude; and becauſe the angles ABK, ABE are giv make the angle NLM equal to ABK, and the angle N equal to ABE. And becauſe the ratio of BF to BD is giv make the ratio of LM to P the ſame with it; and becauſe ratio of the figure BD to BG is given, find this ratio by 53. Dat. and make the ratio of P to Q the ſame. Alſo, beca the ratio of CB to BA is given, make the ratio of Q to R ſame; and take LN equal to R; thro' the point M draw parallel to LN, and complete the parallelogram NLOS; t this is ſimilar to the parallelogram BF.

Becauſe the angle ABK is equal to NLM, and the an ABE to NLO, the angle KBE is equal to MLO; and angles BKE, LMO are equal, becauſe the angle ABK is qual to NLM; therefore, the triangles BKE, LMO are eq angular to one another; wherefore as BE to BK, ſo is LO LM; and becauſe as the figure BF to BD, ſo is the ſtrai line LM to P; and as BD to BG, ſo is P to Q; ex æqu as BF, that is [d] BH, to BG, ſo is LM to Q: But BH is

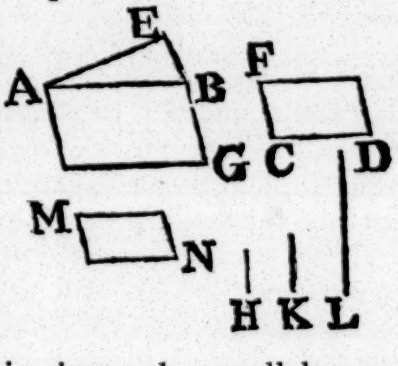
G, as KB to BC; as therefore KB to BC, ſo is LM to Q; d becauſe BE is to BK, as LO to LM; and as BK to BC, is LM to Q; and as BC to BA, ſo Q was made to R; there-re, ex æquali, as BE to BA, ſo is LO to R, that is, to LN; d the angles ABE, NLO are equal; therefore the paralle-gram BF is ſimilar to LS.

PROP. LXX. 62. 78.

IF two ſtraight lines have a given ratio to one another, See N.
and upon one of them be deſcribed a rectilineal figure ven in ſpecies, and upon the other a parallelogram ha-ng a given angle; if the figure have a given ratio to the rallelogram, the parallelogram is given in ſpecies.

Let the two ſtraight lines AB, CD have a given ratio to e another, and upon AB let the figure AEB given in ſpecies deſcribed, and upon CD the parallelogram DF having the ven angle FCD; if the ratio of AEB to DF be given, the pa-llelogram DF is given in ſpecies.

Upon the ſtraight line AB, conceive the parallelogram AG be deſcribed ſimilar, and ſimilarly placed to FD; and becauſe e ratio of AB to CD is given, and upon them are deſcribed e ſimilar rectilineal figures A G, D; the ratio of AG to FD is gi-n[a]; and the ratio of FD to AEB given; therefor[b] the ratio of EB to AG is given; and the angle BG is given, becauſe it is equal to e angle FCD; becauſe therefore e parallelogram AG which has a ven angle ABG is applied to a ſide B of the figure AEB given in ſpe-es, and the ratio of AEB to AG is given, the parallelogram G is given[c] in ſpecies; but FD is ſimilar to AG; therefore D is given in ſpecies.



a 54. Dat.
b 9. Dat.
c 69. Dat.

A parallelogram ſimilar to FD may be found thus; take a aight line H given in magnitude; and becauſe the ratio of e figure AEB to FD is given, make the ratio of H to K the me with it: Alſo, becauſe the ratio of the ſtraight line CD to B is given, find by the 54. Dat. the ratio which the figure D deſcribed upon CD has to the figure AG deſcribed upon B ſimilar to FD; and make the ratio of K to L the ſame ith this ratio: And becauſe the ratios of H to K, and of K to

b 9. Dat. to L are given, the ratio of H to L is given[b]; becauſe therefore, as AEB to FD, ſo is H to K; and as FD to AG, ſo K to L; ex æquali, as AEB to AG, ſo is H to L; therefore the ratio of AEB to AG is given; and the figure AE is given in ſpecies, and to its ſide AB the parallelogram A is applied in the given angle ABG; therefore by the 69. D a parallelogram may be found ſimilar to AG: Let this be t parallelogram MN; MN alſo is ſimilar to FD; for, by the conſtruction, MN is ſimilar to AG, and AG is ſimilar to FD; therefore the parallelogram FD is ſimilar to MN.

81. PROP. LXXI.

IF the extremes of three proportional ſtraight lines ha given ratios to the extremes of other three proportional ſtraight lines; the means ſhall alſo have a given tio to one another: And if one extreme has a given tio to one extreme, and the mean to the mean; likew the other extreme ſhall have to the other a given rat

Let A, B, C be three proportional ſtraight lines, and D, F three other; and let the ratios of A to D, and of C to F given; then the ratio of B to E is alſo given.

Becauſe the ratio of A to D, as alſo of C to F is given, ratio of the rectangle A, C to the rectangle D, F is give
a 67. Dat. b 17. 6. but the ſquare of B is equal[b] to the rectangle A, C; and ſquare of E to the rectangle[b] D, F; therefore the ratio of
c 58. Dat. ſquare of B to the ſquare of E is given; wherefore[c] alſo ratio of the ſtraight line B to E is given.

Next, let the ratio of A to D, and of B to E be given; then the ratio of C to F is alſo given.

A B
D E

Becauſe the ratio of B to E is given, the ratio of
d 54. Dat. the ſquare of B to the ſquare of E is given[d]; therefore[b] the ratio of the rectangle A, C to the rectangle D, F is given; and the ratio of the ſide A to the ſide D is given; therefore the ratio of the other ſide C to
e 65. Dat. the other F is given[e].

COR. And if the extremes of four proportionals have to extremes of four other proportionals given ratios, and one the means a given ratio to one of the means; the other m ſhall have a given ratio to the other mean, as may be ſhewn the ſame manner as in the foregoing Propoſition.

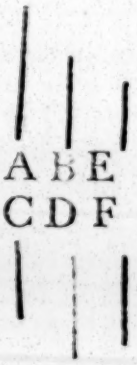
PRO

PROP. LXXII. 82.

IF four ſtraight lines be proportionals; as the firſt is to the ſtraight line to which the ſecond has a given ratio; ſo is the third to a ſtraight line to which the fourth has a given ratio.

Let A, B, C, D be four proportional ſtraight lines, viz. as A to B, ſo C to D; as A is to the ſtraight line to which B has a given ratio, ſo is C to a ſtraight line to which D has a given ratio.

Let E be the ſtraight line to which B has a given ratio, and as B to E, ſo make D to F: The ratio of B to E is given[a], and therefore the ratio of D to F; and becauſe as A to B, ſo is C to D; and as B to E, ſo D to F; therefore, ex æquali, as A to E, ſo is C to F; and E is the ſtraight line to which B has a given ratio, and F that to which D has a given ratio; therefore as A is to the ſtraight line to which B has a given ratio, ſo is C to a line to which D has a given ratio.



a Hyp.

PROP. LXXIII. 83.

IF four ſtraight lines be proportionals; as the firſt is to the ſtraight line to which the ſecond has a given ratio, ſo is a ſtraight line to which the third has a given ratio to the fourth. See N.

Let the ſtraight line A be to B, as C to D; as A to the ſtraight line to which B has a given ratio, ſo is a ſtraight line to which C has a given ratio to D.

Let E be the ſtraight line to which B has a given ratio, and as B to E, ſo make F to C; becauſe the ratio of B to E is given, the ratio of C to F is given: And becauſe A is to B, as C to D; and as B to E, ſo F to C; therefore, ex æquali in proportione perturbata[a], A is to E, as F to D; that is, A is to E which B has a given ratio, as F, to which C has a given ratio, is to D.

a 23. 5.

64. PROP. LXXIV.

IF a triangle has a given obtuſe angle; the exceſs of the ſquare of the ſide which ſubtends the obtuſe angle above the ſquares of the ſides which contain it, ſhall have a given ratio to the triangle.

Let the triangle ABC have a given obtuſe angle ABC; and produce the ſtraight line CB, and from the point A draw AD perpendicular to BC: The exceſs of the ſquare of AC above the ſquares of AB, BC, that is [a], the double of the rectangle contained by DB, BC, has a given ratio to the triangle ABC.

a 12. 2.

Becauſe the angle ABC is given, the angle ABD is alſo given; and the angle ADB is given; wherefore the triangle ABD is given [b] in ſpecies; and therefore the ratio of AD to DB is given: And as AD to DB, ſo is [c] the rectangle AD, BC to the rectangle DB, BC; wherefore the ratio of the rectangle AD, BC to the rectangle DB, BC is given, as alſo the ratio of twice the rectangle DB, BC to the rectangle AD, BC: But the ratio of the rectangle AD, BC to the triangle ABC is given, becauſe it is double [d] of the triangle; therefore the ratio of twice the rectangle DB, BC to the triangle ABC is given [e]; and twice the rectangle DB, BC is the exceſs [a] of the ſquare of AC above the ſquares of AB, BC; therefore the exceſs has a given ratio to the triangle ABC.

b. 43. Dat.
c 1. 6.
d 41. 1.
e. 9. Dat.

And the ratio of this exceſs to the triangle ABC may be found thus; take a ſtraight line EF given in poſition and magnitude; and becauſe the angle ABC is given, at the point F of the ſtraight line EF, make the angle EFG equal to the angle ABC; produce GF, and draw EH perpendicular to FG; then the ratio of the exceſs of the ſquare of AC above the ſquares of AB, BC to the triangle ABC is the ſame with the ratio of quadruple the ſtraight line HF to HE.

Becauſe the angle ABD is equal to the angle EFH, and the angle ADB to EHF, each being a right angle; the triangle ADB is equiangular to EHF; therefore [f] as BD to DA, ſo FH to HE; and as quadruple of BD to DA, ſo is [g] quadruple of FH to HE: But as twice BD is to DA, ſo is [c] twice the rectangle DB, BC to the rectangle AD, BC; and as DB to the half of it, ſo is [h] the rectangle AD, BC to its half triangle

f 4. 6.
g Cor. 4. 5.
h C. 5.

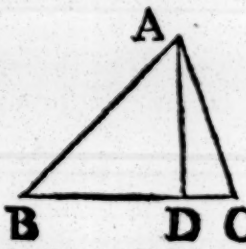
iangle ABC; therefore, ex equali, as twice BD is to the half DA, that is, as quadruple of BD is to DA, that is, as quaruple of FH to HE, ſo is twice the rectangle DB, BC to the iangle ABC.

PROP. LXXV. 65.

F a triangle has a given acute angle; the ſpace, by which the ſquare of the ſide ſubtending the acute gle is leſs than the ſquares of the ſides which contain ſhall have a given ratio to the triangle.

Let the triangle ABC have a given acute angle ABC, and aw AD perpendicular to BC; the ſpace by which the ſquare AC is leſs than the ſquares of AB, BC, that is [a], the double the rectangle contained by CB, BD, has a given ratio to triangle ABC. a. 13. 2.

Becauſe the angles ABD, ADB are each of them given, triangle ABD is given in ſpecies; and therefore the ratio BD to DA is given: And as BD to DA, is the rectangle CB, BD to the rectangle B, AD; therefore the ratio of theſe rectgles is given, as alſo the ratio of twice the ctangle CB, BD to the rectangle CB, AD; t the rectangle CB, AD has a given ratio its half the triangle ABC; therefore [b] the io of twice the rectangle CB, BD to the triangle ABC is gin; and twice the rectangle CB, BD is [a] the ſpace by which ſquare of AC is leſs than the ſquares of AB, BC; therere the ratio of this ſpace to the triangle ABC is given; and ratio may be found as in the preceeding Propoſition.

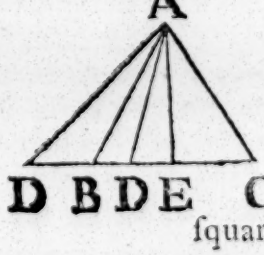


b 9. Dat.

LEMMA,

F from the vertex A of an Iſoſceles triangle ABC, any ſtraight line AD be drawn to the baſe BC; the ſqua of ſide AB is equal to the rectangle BD, DC of the ſegments the baſe together with the ſquare of AD; but if AD be wn to the baſe produced, the ſquare of AD is equal to the ctangle BD, DC together with the ſquare of AB.

Cas. 1. Biſect the baſe BC in E, and n AE which will be perpendicular [a] to ; wherefore the ſquare of AB is equal o the ſquares of AE, EB; but the ſquare EB is equal [c] to the rectangle BD, DC gether with the ſquare of DE; therere the ſquare of AB is equal to the



a. 8. 1.

b. 4 7. 1.

c. 5. 1.

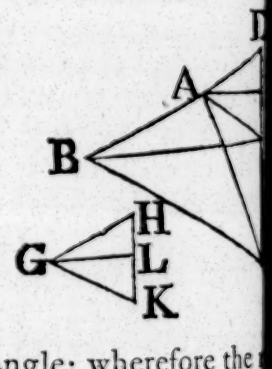
b 47. 1. ſquares of AE, ED, that is, to [b] the ſquare of AD, toget with the rectangle BD, DC; the other caſe is ſhewn in ſame way by 6. 2. Elem.

67. PROP. LXXVI.

IF a triangle have a given angle, the exceſs of the ſqu of the ſtraight line which is equal to the two ſides t contain the given angle, above the ſquare of the th ſide, ſhall have a given ratio to the triangle.

Let the triangle ABC have the given angle BAC, the ex of the ſquare of the ſtraight line which is equal to BA, together above the ſquare of BC, ſhall have a given ratio to triangle ABC.

Produce BA, and take AD equal to AC, join DC produce it to E, and thro' the point B draw BE paralle AC; join AE, and draw AF perpendicular to DC; and cauſe AD is equal to AC, BD is equal to BE; and B drawn from the vertex B of the Iſoſceles triangle DBE, th fore, by the Lemma, the ſquare of BD, that is, of BA AC together, is equal to the rectangle DC, CE together the ſquare of BC; and therefore, the ſquare of BA, AC gether, that is, of BD, is greater than the ſquare of BC by the rectangle DC, CE; and this rectangle has a given ratio to the triangle ABC; becauſe the angle BAC is given, the adjacent angle CAD is given; and each of the angles ADC, DCA is given, for each of them is the half [a] of the given angle BAC; therefore the triangle ADC is given [b] in ſpecies; and AF is drawn from its vertex to the baſe in a given angle; wherefore the of AF to the baſe CD is given [c]; and as CD to AF, ſo is rectangle DC, CE to the rectangle AF, CE; and the rat the rectangle AF, CE to its half [e] the triangle ACE is gi therefore the ratio of the rectangle DC, CE to the tri ACE, that is [f], to the triangle ABC, is given [g]; and the recta DC, CE is the exceſs of the ſquare of BA, AC together a the ſquare of BC; therefore the ratio of this exceſs to th angle ABC is given.



a 5. & 32. 1.
b 43. Dat.
c 50. Dat.
d 1. 6.
e 41. 1.
f 37. 1.
g 9. Dat.

The ratio which the rectangle DC, CE has to the tri ABC is found thus: Take the ſtraight line GH given in

on and magnitude, and at the point G in GH make the angle GK equal to the given angle CAD, and take GK equal to H, join KH, and draw GL perpendicular to it: Then the tio of HK to the half of GL is the ſame with the ratio of e rectangle DC, CE to the triangle ABC; becauſe the angles GK, DAC at the vertices of the Iſoſceles triangles GHK, DC are equal to one another, theſe triangles are ſimilar; and ecauſe GL, AF are perpendicular to the baſes HK, DC, as K to GL, ſo is [h] (DC to AF, and ſo is) the rectangle DC, E to the rectangle AF, CE; but as GL to its half, ſo is the ctangle AF, CE to its half, which is the triangle ACE, or e triangle ABC; therefore, ex æquali, HK is to the half of e ſtraight line GL, as the rectangle DC, CE is to the triangle BC. h. {4. 6. 22. 5.

Cor. And if a triangle have a given angle, the ſpace by hich the ſquare of the ſtraight line which is the difference f the ſides which contain the given angle is leſs than the uare of the third ſide, ſhall have a given ratio to the triangle. his is demonſtrated the ſame way as the preceeding Propoſi-on, by help of the ſecond caſe of the Lemma.

PROP. LXXVII. I.

F the perpendicular drawn from a given angle of a tri- See N.
angle to the oppoſite ſide, or baſe, has a given ratio the baſe; the triangle is given in ſpecies.

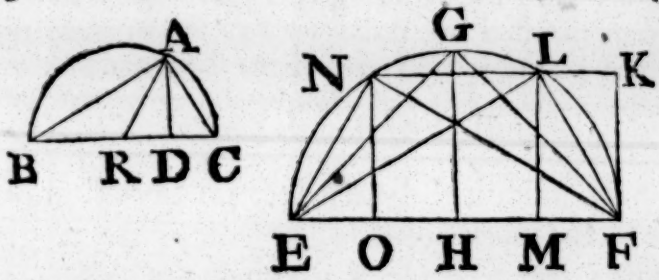
Let the triangle ABC have the given angle BAC, and let e perpendicular AD drawn to the baſe BC, have a given ra-o to it; the triangle ABC is given in ſpecies.

If ABC be an Iſoſceles triangle, it is evident [a] that if any a 5. & 32. 1.

ne of its angles be given, the reſt are alſo given; and there-re the triangle is given in ſpecies, without the conſideration f the ratio of the perpendicular to the baſe, which in this caſe is ven by Prop. 50.

But when ABC is not an Iſoſceles triangle, take any ſtraight ne EF given in poſition and magnitude, and upon it deſcribe

the ſegment of a circle EGF containing an angle equal to t
given angle BAC; draw GH biſecting EF at right angles, a
join EG, GF: Then, ſince the angle EGF is equal to the an
BAC, and that EGF is an Iſoſceles triangle and ABC is n
the angle FEG is not equal to the angle CBA: Draw EL m
king the angle FEL equal to the angle CBA; join FL, a
draw LM perpendicular to EF; then, becauſe the triangles EL
BAC are equiangular, as alſo are the triangles MLE, DA
as ML to LE, ſo is DA to AB; and as LE to EF, ſo is AB to B
wherefore, ex æquali, as LM to EF, ſo is AD to BC; and
cauſe the ratio of AD to BC is given, therefore the ratio of L
b 2. Dat. to EF is given; and EF is given, wherefore [b] LM alſo is giv
Complete the parallelogram LMFK; and becauſe LM is giv
FK is given in magnitude; it is alſo given in poſition,
c 30. Dat. the point F is given, and conſequently, [c] the point K; and
cauſe through K the ſtraight line KL is drawn parallel to
d 31. Dat. which is given in poſition, therefore [d] KL is given in poſitio



and the circumference ELF is given in poſition; therefore
e 28. Dat. point L is given [e]. And becauſe the points L, E, F are giv
f 29. Dat. the ſtraight lines LE, EF, FL are given [f] in magnitude; th
g 42. Dat. fore the triangle LEF is given in ſpecies [g]; and the trian
ABC is ſimilar to LEF, wherefore alſo ABC is given in ſpec

Becauſe LM is leſs than GH, the ratio of LM to EF,
is, the given ratio of AD to BC, muſt be leſs than the rati
GH to EF, which the ſtraight line, in a ſegment of a ci
containing an angle equal to the given angle, that biſects
baſe of the ſegment at right angles, has unto the baſe.

Cor. 1. If two triangles, ABC, LEF have one angle B
equal to one angle ELF, and if the perpendicular AD be to
baſe BC, as the perpendicular LM to the baſe EF; the trian
ABC, LEF are ſimilar.

Deſcribe the circle EGF about the triangle ELF, and d
LN parallel to EF, join EN, NF, and draw NO perpendi
lar to EF; becauſe the angles ENF, ELF are equal, and

the angle EFN is equal to the alternate angle FNL, that is, to the angle FEL in the ſame ſegment; therefore the triangle NEF is ſimilar to LEF; and in the ſegment EGF there can be no other triangle upon the baſe EF, which has the ratio of its perpendicular to that baſe the ſame with the ratio of LM or NO to EF, becauſe the perpendicular muſt be greater or leſs than LM or NO; but, as has been ſhewn in the preceeding demonſtration, a triangle ſimilar to ABC can be deſcribed in the ſegment EGF upon the baſe EF, and the ratio of its perpendicular to the baſe is the ſame, as was there ſhewn, with the ratio of AD to BC, that is, of LM to EF; therefore, that triangle muſt be either LEF, or NEF, which therefore are ſimilar to the triangle ABC.

Cor. 2. If a triangle ABC has a given angle BAC, and if the ſtraight line AR drawn from the given angle to the oppoſite ſide BC, in a given angle ARC, has a given ratio to BC; the triangle ABC is given in ſpecies.

Draw AD perpendicular to BC; therefore the triangle ARD is given in ſpecies; wherefore, the ratio of AD to AR is given; and the ratio of AR to BC is given, and conſequently[h], the ratio of AD to BC is given; and the triangle ABC is therefore given in ſpecies[i]. h 9. Dat. i 77. Dat.

Cor. 3. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if ſtraight lines drawn from theſe angles to the baſes, making with them given and equal angles, have the ſame ratio to the baſes, each to each; then the triangles are ſimilar; for, having drawn perpendiculars to the baſes from the equal angles, as one perpendicular is to its baſe, ſo is the other to its baſe[k]; wherefore, by Cor. 1. the triangles are ſimilar. k, {4. 6. / 22. 5.}

A triangle ſimilar to ABC may be found thus; having deſcribed the ſegment EGF and drawn the ſtraight line GH as was directed in the Propoſition, find FK which has to EF the given ratio of AD to BC; and place FK at right angles to EF from the point F; then becauſe, as has been ſhewn, the ratio of AD to BC, that is, of FK to EF, muſt beleſs than the ratio of GH to EF; therefore FK is leſs than GH; and conſequently, the parallel to EF drawn thro' the point K, muſt meet the circumference of the ſegment in two points: Let L be either of them, and join EL, LF, and draw LM perpendicular to EF; then, becauſe the angle BAC is equal to the angle ELF, and that AD is to BC, as KF, that is, LM to EF, the triangle ABC is ſimilar to the triangle LEF, by Cor. 1.

80. PROP. LXXVIII.

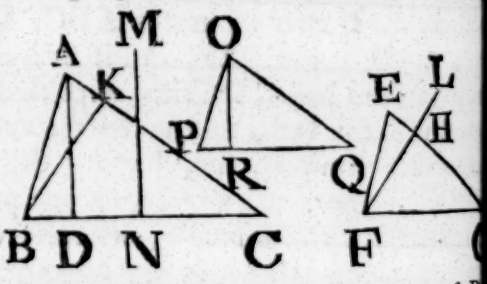
IF a triangle have one angle given, and if the ratio of the rectangle of the ſides which contain the given angle to the ſquare of the third ſide be given; the triangle is given in ſpecies.

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the ſquare of BC be given; the triangle ABC is given in ſpecies.

From the point A, draw AD perpendicular to BC; the rectangle AD, BC has a given ratio to its half [a] the triangle ABC, and becauſe the angle BAC is given, the ratio of the triangle ABC to the rectangle BA, AC is given [b]; and, by the hypotheſis, the ratio of the rectangle BA, AC to the ſquare of BC given; therefore, [c] the ratio of the rectangle AD, BC to the ſquare of BC, that is [d] the ratio of the ſtraight line AD to BC, given; wherefore the triangle ABC is given in ſpecies [e].

a 41. 1.
b Cor. 62. Dat.
c 9. Dat.
d 1. 6.
e 77. Dat.

A triangle ſimilar to ABC may be found thus; take a ſtraight line EF given in poſition and magnitude, and make the angle FEG equal to the given angle BAC, and draw FH perpendicular to EG, and BK perpendicular to AC; therefore the triangles ABK, EFH are ſimilar, and the rectangle AD, BC, or the rectangle BK, AC, which is equal to it, is to the rectangle BA, AC, as the ſtraight line BK to BA, that is, as FH to FE. Let the given ratio of the rectangle BA, AC to the ſquare of BC be the ſame with the ratio of the ſtraight line EF to FL; therefore, ex æquali, the ratio of the rectangle AD, BC to the ſquare of BC, that is, the ratio of the ſtraight line AD to BC is the ſame with the ratio of HF to FL; and becauſe AD is not greater than the ſtraight line MN in the ſegment of the circle deſcribed about the triangle ABC, which biſects BC at right angles; the ratio of AD to BC, that is, of HF to FL, muſt not be greater than the ratio of MN to BC. Let it be ſo, and, by the 77. Dat. find a triangle OPQ which has one of its angles POQ equal to the given angle BAC, and the ratio of the perpendicular OR drawn from that angle to the baſe PQ the ſame with the ratio of HF to FL; then the triangle ABC is ſimilar



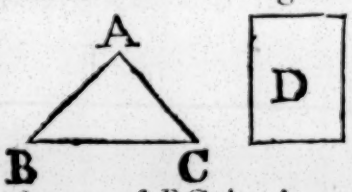
OP

PQ, becauſe, as has been ſhewn, the ratio of AD to BC the ſame with the ratio of (HF to FL, that is, by the conſtruction, with the ratio of) OR to PQ; and the angle BAC equal to the angle POQ; therefore the triangle ABC is milar[f] to the triangle POQ. f. 1. Cor. 77. Dat.

Otherwiſe,

Let the triangle ABC have the given angle BAC, and let he ratio of the rectangle BA, AC to the ſquare of BC be iven; the triangle ABC is given in ſpecies.

Becauſe the angle BAC is given, the exceſs of the ſquare both the ſides BA, AC together above the ſquare of the ird ſide BC has a given[a] ratio to the triangle ABC. Let the a.76.Dat. gure D be equal to this exceſs; therefore the ratio of D to he triangle ABC is given; and the ratio of the triangle ABC the rectangle BA, AC is given[b], becauſe BAC is a given b.Cor.62 Dat. ngle; and the rectangle BA, AC has given ratio to the ſquare of BC; herefore[c] the ratio of D to the uare of BC is given; and, by com-ſition[d], the ratio of the ſpace D gether with the ſquare of BC to the ſquare of BC is given; t D together with the ſquare of BC is equal to the ſquare f both BA and AC together; therefore the ratio of the uare of BA, AC together to the ſquare of BC is given; and he ratio of BA, AC together to BC is therefore given[e]; and e.59.Dat. he angle BAC is given, wherefore[f] the triangle ABC is given f.48.Dat. n ſpecies.



c. 10.Dat.

d. 7. Dat.

The compoſition of this which depends upon thoſe of the 6. and 48. Propoſitions is more complex than the preceedng compoſition which depends upon that of Prop. 77. which s eaſy.

PROP. LXXIX. K.

IF a triangle have a given angle, and if the ſtraight line See N. drawn from that angle to the baſe, making a given ngle with it, divides the baſe into ſegments which have given ratio to one another; the triangle is given in pecies.

Let the triangle ABC have the given angle BAC, and let he ſtraight line AD drawn to the baſe BC making the given ngle ADB, divide BC into the ſegments BD, DC which have a given

a given ratio to one another; the triangle ABC is given in ſpecie

a. 5. 4. Deſcribe[a] the circle BAC about the triangle, and from i center E, draw EA, EB, EC, ED; becauſe the angle BAC

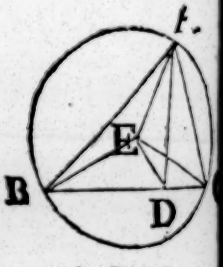
b. 20. 3. given, the angle BEC at the center, which is the double[b] of is given. And the ratio of BE to EC is given, becauſe th

c. 44. Dat. are equal to one another; therefore[c] the triangle BEC given in ſpecies, and the ratio of EB to BC given; al

d. 7. Dat. the ratio of CB to BD is given[d], becauſe the ratio of BD

e. 9. Dat. DC is given; therefore the ratio of EB to BD is given[e], a the angle EBC is given, wherefore the triangle EBD is give in ſpecies, and the ratio of EB, that is, of EA to ED, is ther fore given; and the angle EDA is given, becauſe each of t angles BDE, BDA is given; therefore the triangle AED

f. 47. Dat. given[f] in ſpecies, and the angle AED given; alſo the angle DEC is given, becauſe each of the angles BED, BEC is given; therefore the angle AEC is given, and the ratio of EA to EC, which are equal, is given; and the triangle AEC is therefore given[c] in ſpecies, and the angle ECA given; and the angle ECB is given, wherefore the angle ACB is given, and the angle BAC is a



g. 43. Dat. given; therefore[g] the triangle ABC is given in ſpecies.

A triangle ſimilar to ABC may be found, by taking a ſtraig line given in poſition and magnitude, and dividing it in t given ratio which the ſegments BD, DC are required to ha to one another; then, if upon that ſtraight line a ſegment of circle be deſcribed containing an angle equal to the given ang BAC, and a ſtraight line be drawn from the point of diviſion an angle equal to the given angle ADB, and from the poi where it meets the circumference, ſtraight lines be drawn the extremity of the firſt line, theſe together with the firſt li ſhall contain a triangle ſimilar to ABC, as may eaſily be ſhew

The Demonſtration may be alſo made in the manner of th of the 77. Prop. and that of the 77. may be made in the mann of this.

PROP. LXXX.

L. IF the ſides about an angle of a triangle have a giv ratio to one another, and if the perpendicular draw from that angle to the baſe has a given ratio to the baſe the triangle is given in ſpecies.

L

Let the ſides BA, AC, about the angle BAC of the triangle ABC have a given ratio to one another, and let the perpendicular AD have a given ratio to the baſe BC; the triangle ABC is given in ſpecies.

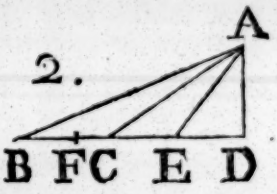
Firſt, let the ſides AB, AC be equal to one another, therefore the perpendicular AD biſects[a] the baſe BC; and the ratio of AD to BC, and therefore to its half DB, is given; and the angle ADB is given; wherefore the triangle * ABD, and conſequently the triangle ABC, is given[b] in ſpecies.

a. 26. 1.

* 43. Dat.

b. 44. Dat.

But let the ſides be unequal, and BA be greater than AC; and make the angle CAE equal to the angle ABC; becauſe the angle AEB is common to the triangles AEB, CEA, they are ſimilar; therefore as AB to BE, ſo is CA to AE, and, by permutation, as BA to AC, ſo is BE to EA, and ſo is EA to EC; and the ratio of BA to AC is given, therefore the ratio of BE to EA, and the ratio of EA to EC, as alſo the ratio of BE to EC, is given[c]; wherefore the ratio of EB to BC is given[d]; and the ratio of AD to BC is given by the hypotheſis, therefore[e] the ratio of AD to BE is given; and the ratio of BE to EA was ſhewn to be given; wherefore the ratio of AD to AE is given, and ADE is a right angle, therefore the triangle ADE is given[e] in ſpecies, and the angle AEB given; the ratio of BE to EA is likewiſe given, therefore[b] the triangle ABE is given in ſpecies, and conſequently the angle EAB, as alſo the angle ABE, that is, the angle CAE, is given; therefore the angle BAC is given, and the angle ABC being alſo given, the triangle ABC is given[f] in ſpecies.



c. 9. Dat.

d. 6. Dat.

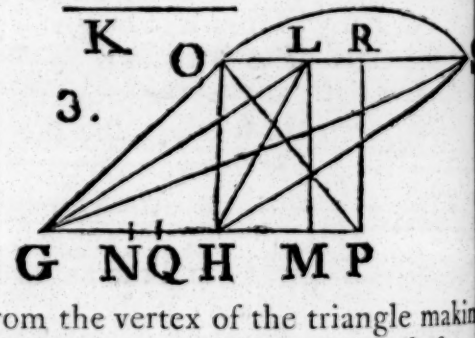
e. 46. Dat.

f. 43. Dat.

How to find a triangle which ſhall have the things which are mentioned to be given in the Propoſition, is evident in the firſt caſe; and to find it the more eaſily in the other caſe, it is to be obſerved that, if the ſtraight line EF equal to EA be placed in EB towards B, the point F divides the baſe BC into the ſegments BF, FC which have to one another the ratio of the ſides BA, AC; becauſe BE, EA, or EF, and EC were ſhewn to be proportionals, therefore * BF is to FC, as BE to EF, or EA, that is, as BA to AC; and AE cannot be leſs than the altitude of the triangle ABC, but it may be equal

* 19. 5.

equal to it; which if it be, the triangle, in this caſe, as al
the ratio of the ſides, may be thus found: Having given th
ratio of the perpendicular to the baſe, take the ſtraight lin
GH given in poſition and magnitude, for the baſe of the tri
angle to be found; and let the given ratio of the perpendicu
lar to the baſe be that of the ſtraight line K to GH, that i
let K be equal to the perpendicular; and ſuppoſe GLH to b
the triangle which is to be found, therefore having made th
angle HLM equal to LGH, it is required that LM b
perpendicular to GM and equal to K; and becauſe GM, MI
MH are proportionals, as was ſhewn of BE, EA, EC, th
rectangle GMH is equal to the ſquare of ML. Add the com
mon ſquare of NH, (having biſected GH in N), and the ſquar
g. 6. 2. of NM is equal [g] to the ſquares of the given ſtraight lines NI
and ML, or K; therefore the ſquare of NM, and its ſid
NM, is given, as alſo the point M, viz. by taking the ſtraigh
line NM the ſquare of which is equal to the ſquares of NH
ML. Draw ML equal to K, at right angles to GM; and be
cauſe ML is given in poſition and magnitude, therefore th
point L is given; join LG, LH; then the triangle LGH
that which was to be found, for the ſquare of NM is equal t
the ſquares of NH and ML, and taking away the commo
ſquare of NH, the rect-
angle GMH is equal [g] to
the ſquare of ML; there-
fore as GM to ML, ſo is
ML to MH, and the tri-
.h 6. 6. angle LGM is [h] therefore
equiangular to HLM, and
the angle HLM equal to
the angle LGM, and the



ſtraight line LM, drawn from the vertex of the triangle makin
the angle HLM equal to LGH, is perpendicular to the baſe an
equal to the given ſtraight line K, as was required; and th
ratio of the ſides GL, LH is the ſame with the ratio of GM t
ML, that is, with the ratio of the ſtraight line which is mad
up of GN the half of the given baſe and of NM, the ſquare
which is equal to the ſquares of GN and K, to the ſtraigh
line K.

And whether this ratio of GM to ML is greater or le
than the ratio of the ſides of any other triangle upon the ba
GH, and of which the altitude is equal to the ſtraight line K
th

that is, the vertex of which is in the parallel to GH drawn thro' the point L, may be thus found. Let OGH be any such triangle, and draw OP making the angle HOP equal to the angle OGH; therefore, as before, GP, PO, PH are proportionals, and PO cannot be equal to LM, becauſe the rectangle GPH would be equal to the rectangle GMH, which is impoſſible; for the point P cannot fall upon M, becauſe O would then fall on L; nor can PO be leſs than LM, therefore it is greater; and conſequently the rectangle GPH is greater than the rectangle GMH, and the ſtraight line GP greater than GM: Therefore the ratio of GM to MH is greater than the ratio of GP to PH, and the ratio of the ſquare of GM to the ſquare of ML is therefore [i] greater than the ratio of the ſquare of GP to the ſquare of PO, and the ratio of the ſtraight line GM to ML greater than the ratio of GP to PO. But as GM to ML, ſo is GL to LH; and as GP to PO, ſo is GO to OH; therefore the ratio of GL to LH is greater than the ratio of GO to OH; wherefore the ratio of GL to LH is the greateſt of all others; and conſequently the given ratio of the greater ſide to the leſs muſt not be greater than this ratio. i. 2. Cor. 20. 6.

But if the ratio of the ſides be not the ſame with this greateſt ratio of GM to ML, it muſt neceſſarily be leſs than it: Let any leſs ratio be given, and the ſame things being ſuppoſed, viz. that GH is the baſe, and K equal to the altitude of the triangle, it may be found as follows. Divide GH in the point Q, ſo that the ratio of GQ to QH may be the ſame with the given ratio of the ſides; and as GQ to QH, ſo make GP to PQ, and ſo will [f] PQ be to PH; wherefore the ſquare of GP is to the ſquare of PQ, as [i] the ſtraight line GP to PH: And becauſe GM, ML, MH are proportionals, the ſquare of GM is to the ſquare of ML, as [i] the ſtraight line GM to MH: But the ratio of GQ to QH, that is, the ratio of GP to PQ, is leſs than the ratio of GM to ML; and therefore the ratio of the ſquare of GP to the ſquare of PQ is leſs than the ratio of the ſquare of GM to that of ML; and conſequently the ratio of the ſtraight line GP to PH is leſs than the ratio of GM to MH; and, by diviſion, the ratio of GH to HP is leſs than that of GH to HM; wherefore [k] the ſtraight line HP is greater than HM, and the rectangle GPH, that is, the ſquare of PQ, greater than the rectangle GMH, that is, than the ſquare f. 19. 5. k. 10. 5.

ſquare of ML, and the ſtraight line PQ is therefore greate
than ML. Draw LR parallel to GP, and from P draw PR a
right angles to GP: Becauſe PQ is greater than ML, or PR
the circle deſcribed from the center P, at the diſtance PQ
muſt neceſſarily cut LR in two Points; let theſe be O, S, an
join OG, OH; SG, SH; each of the triangles OGH, SGH
have the things mentioned to be given in the Propoſition
Join OP, SP; and becauſe as GP to PQ, or PO, ſo is PO
to PH, the triangle OGP is equiangular to HOP; as, there
fore, OG to GP, ſo is HO to OP, and, by permutation, a
GO to OH, ſo is GP to PO, or PQ, and ſo is GQ to QH.
Therefore the triangle OGH has the ratio of its ſides GO, OH
the ſame with the given ratio of GQ to QH; and the perpen
dicular has to the baſe the given ratio of K to GH, becauſe
the perpendicular is equal to LM, or K: The like may be
ſhewn in the ſame way of the triangle SGH.

This conſtruction by which the triangle OGH is found, i
ſhorter than that which would be deduced from the demon
ſtration of the Datum; by reaſon that the baſe GH is given
in poſition and magnitude, which was not ſuppoſed in the
demonſtration: The ſame thing is to be obſerved in the nex
Propoſition.

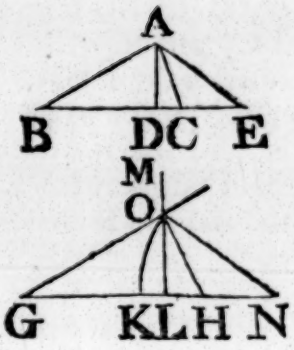
M. PROP. LXXXI.

IF the ſides about an angle of a triangle be unequa
and have a given ratio to one another, and if the
perpendicular from that angle to the baſe divides it into
ſegments that have a given ratio to one another; the
triangle is given in ſpecies.

Let ABC be a triangle, the ſides of which about the angl
BAC are unequal, and have a given ratio to one another
and let the perpendicular AD to the baſe BC divide it into
the ſegments BD, DC which have a given ratio to one another
the triangle ABC is given in ſpecies.

Let AB be greater than AC, and make the angle CA
equal to the angle ABC; and becauſe the angle AEB is com
a. 4. 6. mon to the triangles ABE, CAE, they are [a] equiangular t
one another: Therefore as AB to BE, ſo is CA to AE, and
b

y permutation, as AB to AC, ſo BE to
A, and ſo is EA to EC: But the ratio of
A to AC is given, therefore the ratio
BE to EA, as alſo the ratio of EA to
C, is given; wherefore [b] the ratio of
E to EC, as alſo [c] the ratio of EC to
B is given: And the ratio of BC to CD
given [d], becauſe the ratio of BD to
C is given; therefore [b] the ratio of EC
CD is given, and conſequently [d] the
tio of DE to EC: And the ratio of EC
EA was ſhewn to be given, therefore [b] the ratio of DE to EA
given: And ADE is a right angle, wherefore [e] the trian- e 46. Dat.
e ADE is given in ſpecies, and the angle AED given: And the
tio of CE to EA is given, therefore [f] the triangle AEC is gi- f 44. Dat.
en in ſpecies, and conſequently the angle ACE is given, as
ſo the adjacent angle ACB. In the ſame manner, becauſe the
tio of BE to EA is given, the triangle BEA is given in ſpe-
es, and the angle ABE is therefore given: And the angle
CB is given; wherefore the triangle ABC is given [g] in ſpe- g 43 Dat.
es.



b. 9. Dat.

c. Cor. 6. Dat.

d. 7. Dat.

But the ratio of the greater ſide BA to the other AC muſt
e leſs than the ratio of the greater ſegment BD to DC: Be-
auſe the ſquare of BA is to the ſquare of AC, as the ſquares
f BD, DA to the ſquares of DC, DA; and the ſquares of
D, DA have to the ſquares of DC, DA a leſs ratio than
e ſquare of BD has to the ſquare of DC †, becauſe the
uare of BD is greater than the ſquare of DC; therefore
e ſquare of BA has to the ſquare of AC a leſs ratio than
e ſquare of BD has to that of DC: And conſequently the
tio of BA to AC is leſs than the ratio of BD to DC.

This being premiſed, a triangle which ſhall have the things
entioned to be given in the Propoſition, and to which the
iangle ABC is ſimilar, may be found thus: Take a ſtraight
ne GH given in poſition and magnitude, and divide it in K,
that the ratio of GK to KH may be the ſame with the given
tio of BA to AC: Divide alſo GH in L, ſo that the ratio
of

† If A be greater than B, and C any
rd magnitude; then A and C together
e to B and C together a leſs ratio than
has to B.
Let A be to B as C to D, and becauſe
is greater than B, C is greater than D:
But as A is to B, ſo A and C to B and D;
and A and C have to B and C a leſs ratio
than A and C have to B and D, becauſe C
is greater than D, therefore A and C have
to B and C a leſs ratio than A to B.

of GL to LH may be the ſame with the given ratio of BD DC, and draw LM at right angles to GH: And becauſe ratio of the ſides of a triangle is leſs than the ratio of the ſe ments of the baſe, as has been ſhewn, the ratio of GK to K is leſs than the ratio of GL to LH; wherefore the point muſt fall betwixt K and H: Alſo make as GK to KH, ſo G
h. 19. 5. to NK, and ſo ſhall [h] NK be to NH. And from the center at the diſtance NK, deſcribe a circle, and let its circumferen meet LM in O, and join OG, OH; then OGH is the t angle which was to be deſcribed: Becauſe GN is to NK, NO, as NO to NH, the triangle OGN is equiangular to HO therefore as OG to GN, ſo is HO to ON, and, by permu tion, as GO to OH, ſo is GN to NO, or NK, that is, as G to KH, that is, in the given ratio of the ſides, and, by conſtruction, GL, LH have to one another the given ratio the ſegments of the baſe.

60. PROP. LXXXII.

IF a parallelogram given in ſpecies and magnitude increaſed or diminiſhed by a gnomon given in ma nitude; the ſides of the gnomon are given in mag tude.

Firſt, let the parallelogram AB given in ſpecies and m nitude be increaſed by the given gnomon ECBDFG; e of the ſtraight lines CE, DF is given.

Becauſe AB is given in ſpecies and magnitude, and t the gnomon ECBDFG is given, therefore the whole ſp
a { 2 Def. 2. and 24. 6. AG is given in magnitude: But AG is alſo given in ſpec becauſe it is ſimilar [a] to AB; therefore the ſides of AG
b. 60 Dat. given [b], each of the ſtraight lines AE, AF is therefore given; and each of the ſtraight lines CA, AD is given [b], therefore each
c. 4. Dat. of the remainders EC, DF is given [c].

Next, let the parallelogram AG, given in ſpecies and magnitude, be diminiſhed by the given gnomon ECBDFG; each of the ſtraight lines CF, DE is given.

Becauſe the parallelogram AG is given, as alſo its gnomon ECBDFG, the remaining ſpace AB is give magnitu

but it is also given in species; because it is similar[a] to AG; therefore[b] its sides CA, AD are given, and each of the straight lines EA, AF is given; therefore EC, DF are each of them given.

a { 2. Def. 2. and 24. 5. }
b. 60. Dat.

The gnomon and its sides CE, DF may be found thus in the first case. Let H be the given space to which the gnomon must be made equal, and find[d] a parallelogram similar to AB and equal to the figures AB and H together, and place its sides AE, AF from the point A, upon the straight lines AC, AD, and complete the parallelogram AG, which is about the same diameter[e] with AB; because therefore AG is equal to both AB and H, take away the common part AB, the remaining gnomon ECBDFG is equal to the remaining figure H; therefore a gnomon equal to H, and its sides CE, DF are found: And in like manner they may be found in the other case, in which the given figure H must be less than the figure AE from which it is to be taken.

d. 25. 6.

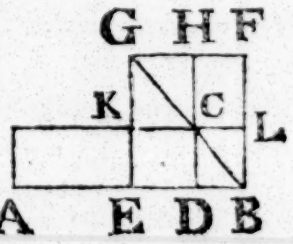
e. 26. 6.

PROP. LXXXIII. 58.

IF a parallelogram equal to a given space be applied to a given straight line, deficient by a parallelogram given in species; the sides of the defect are given.

Let the parallelogram AC equal to a given space be applied to the given straight line AB, deficient by the parallelogram DCL given in species; each of the straight lines CD, DB are given.

Bisect AB in E; therefore EB is given in magnitude, upon EB describe[a] the parallelogram EF similar to DL and similarly placed; therefore EF is given in species, and is about the same diameter[b] with DL; let BCG be the diameter, and construct the figure; therefore, because the figure EF given in species is described upon the given straight line EB, EF is given[c] in magnitude, and the gnomon ELH is equal[d] to the given figure AC; therefore[e] since EF is diminished by the given gnomon ELH, the sides EK, FH of the gnomon are given, but EK is equal to DC, and FH to DB; wherefore CD, DB are each of them given.



a. 18. 6.

b. 26. 6.

c. 56. Dat.

d. 36. and 43. 1.

e. 82. Dat.

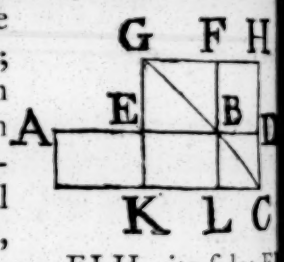
This Demonſtration is the analyſis of the problem in t 28. Prop. of Book 6. the conſtruction and demonſtration which Propoſition is the compoſition of the analyſis; and b cauſe the given ſpace AC or its equal the gnomon ELH is be taken from the figure EF deſcribed upon the half of A ſimilar to BC, therefore AC muſt not be greater than EF, is ſhewn in the 27. Prop. B. 6.

59. PROP. LXXXIV.

IF a parallelogram equal to a given ſpace be applied a given ſtraight line, exceeding by a parallelogra given in ſpecies; the ſides of the exceſs are given.

Let the parallelogram AC equal to a given ſpace be appli to the given ſtraight line AB, exceeding by the parallelogra BDCL given in ſpecies; each of the ſtraight lines CD, D are given.

Biſect AB in E; therefore EB is given in magnitude: Up
a. 18. 6. EB deſcribe [a] the parallelogram EF ſimilar to LD, and ſimila ly placed; therefore EF is given in ſpecies, and is about t
b. 26. 6. ſame diameter [b] with LD. Let CBG be the diameter, and conſtruct the figure; therefore, becauſe the figure EF given in ſpecies is deſcribed upon the given ſtraight line EB, EF is given in magni-
c. 56. Dat. tude [c], and the gnomon ELH is equal
d. 36. and to the given figure [d] AC; wherefore,
43. 1. ſince EF is increaſed by the given gnomon ELH, its ſides E
e. 82. Dat. FH are given [e]; but EK is equal to CD, and FH to BD; ther fore CD, DB are each of them given.



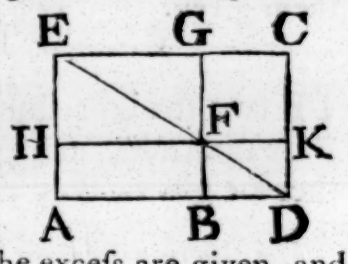
This demonſtration is the analyſis of the Problem in t 29. Prop. Book 6. the conſtruction and demonſtration which is the compoſition of the analyſis.

COR. If a parallelogram given in ſpecies be applied to given ſtraight line, exceeding by a parallelogram equal to given ſpace; the ſides of the parallelogram are given.

Let the parallelogram ADCE given in ſpecies be applied the given ſtraight line AB exceeding by the parallelogra BDCG equal to a given ſpace; the ſides AD, DC of the p rallelogram are given.

Dra

Draw the diameter DE of the parallelogram AC, and construct the figure. Becauſe the parallelogram AK is equal [a] to BC which is given, therefore AK is given; and BK is ſimilar [b] to AC, therefore BK is given in ſpecies. And ſince the parallelogram AK given in magnitude is applied to the given ſtraight line AB, exceeding by the parallelogram BK given in ſpecies, therefore, by this Propoſition, BD, DK the ſides of the exceſs are given, and the ſtraight line AB is given; therefore the whole AD, as alſo DC, to which it has a given ratio, is given.



a. 43. 1.

b. 24. 6.

PROB.

To apply a parallelogram ſimilar to a given one to a given ſtraight line AB, exceeding by a parallelogram equal to a given ſpace.

To the given ſtraight line AB apply [c] the parallelogram AK equal to the given ſpace, exceeding by the parallelogram BK ſimilar to the one given. Draw DF the diameter of BK, and thro' the point A draw AE parallel to BF meeting DF produced in E, and complete the parallelogram AC.

c. 29. 6.

The parallelogram BC is equal [a] to AK, that is, to the given ſpace; and the parallelogram AC is ſimilar [b] to BK; therefore the parallelogram AC is applied to the ſtraight line AB ſimilar to the one given and exceeding by the parallelogram BC which is equal to the given ſpace.

PROP. LXXXV. 84.

If two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the difference of the ſtraight lines be given, they ſhall each of them be given.

Let AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let the exceſs of BC above AB be given; each of the ſtraight lines AB, BC is given.

Let DC be the given exceſs of BC above BA, therefore the remainder BD is equal to BA. Complete the parallelogram AD; and becauſe AB is equal to BD, the ratio of AB to BD is given; and the angle ABD is given, therefore the parallelogram AD is given in ſpecies; and becauſe the given parallelogram AC is applied to the given ſtraight line DC, exceeding by the parallelogram AD given in ſpecies, the ſides of the exceſs are given [a]; there-

a 84. Dat.

therefore BD is given; and DC is given, wherefore the whole BC is given; and AB is given, therefore AB, BC are each of them given.

85. PROP. LXXXVI.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if both of them together be given, they ſhall each of them be given.

Let the two ſtraight lines AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let AB, BC together be given; each of the ſtraight lines AB, BC is given.

Produce CB, and make BD equal to BA, and complete the parallelogram ABDE. Becauſe DB is equal to BA, and the angle ABD given, becauſe the adjacent angle ABC is given, the parallelogram AD is given in ſpecies: And becauſe AB, BC together are given, and AB is equal to BD; therefore DC is given: And becauſe the given parallelogram AC is applied to the given ſtraight line DC, deficient by the parallelogram AD given in

a.83.Dat. ſpecies, the ſides AB, BD of the defect are given[a]; and DC is given, wherefore the remainder BC is given; and each of the ſtraight lines AB, BC is therefore given.

87. PROP. LXXXVII.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the exceſs of the ſquare of the greater above the ſquare of the leſſer be given, each of the ſtraight lines ſhall be given.

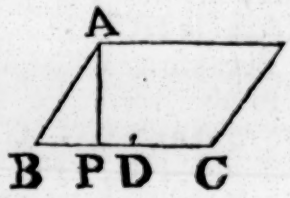
Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC; if the exceſs of the ſquare of BC above the ſquare of BA be given; AB and BC are each of them given.

Let the given exceſs of the ſquare of BC above the ſquare of BA be the rectangle CB, BD; take this from the ſquare of BC, the remainder, which is

a 2.2. [a] the rectangle BC, CD equal to the ſquare of AB; and becauſe the angle ABC is given, the parallelogram AC is given, the ratio of the rectangle

b.62.Dat. of the ſides AB, BC to the parallelogram AC is given[b]; AC is given, therefore the rectangle AB, BC is given; the rectangle CB, BD is given therefore the ratio of the

ngle CB, BD to the rectangle AB, BC, that is, [c] the ratio of the c. 1. 6.
traight line DB to BA is given; therefore [d] the ratio of the d 54. Dat.
quare of DB to the ſquare of BA is gi-
ven: And the ſquare of BA is equal to
he rectangle BC, CD; wherefore the ra-
io of the rectangle BC, CD to the ſquare
f BD is given, as alſo the ratio of four
imes the rectangle BC, CD to the ſquare
f BD; and, by compoſition [e], the ratio of four times the rect- e. 7. Dat
ngle BC, CD together with the ſquare of BD to the ſquare
f BD is given: But four times the rectangle BC, CD toge-
her with the ſquare of BD is equal [f] to the ſquare of the ſtraight f. 8. 2.
ines BC, CD taken together; therefore the ratio of the ſquare
f BC, CD together to the ſquare of BD is given; wherefore
the ratio of the ſtraight line BC together with CD to BD is g 58. Dat.
iven: And, by compoſition, the ratio of BC together with
D and DB, that is, the ratio of twice BC to BD, is given;
herefore the ratio of BC to BD is given, as alſo [c] the ratio of
he ſquare of BC to the rectangle CB, BD: But the rectangle
B, BD is given, being the given exceſs of the ſquares of BC,
A; therefore the ſquare of BC, and the ſtraight line BC is
iven: And the ratio of BC to BD, as alſo of BD to BA has
een ſhewn to be given; therefore [h] the ratio of BC to BA is h. 9. Dat.
iven; and BC is given, wherefore BA is given.



The preceeding demonſtration is the analyſis of this Problem, viz.

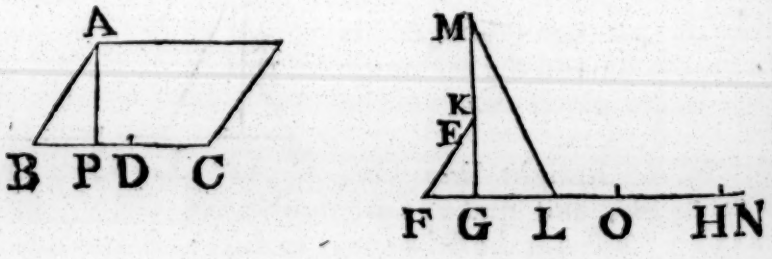
A parallelogram AC which has a given angle ABC being given in magnitude, and the exceſs of the ſquare of BC one of its ſides above the ſquare of the other BA being given; to find the ſides: And the compoſition is as follows.

Let EFG be the given angle to which the angle ABC is required to be equal, and from any point E in FE, draw EG perpendicular to FG; let the rect-
angle EG, GH be the given ſpace
to which the parallelogram AC is
to be made equal; and the rectangle
HG, GL be the given exceſs of the
ſquares of BC, BA.

Take, in the ſtraight line GE, GK equal to FE, and make GM double of GK; join ML, and in GL produced, take LN equal to LM: Biſect GN in O, and between GH, GO find a mean proportional BC: As OG to GL, ſo make CB to BD; and make the angle CBA equal

to CFE, and as LG to GK ſo make DB to BA; and complete the parallelogram AC: AC is equal to the rectangle EG, GH, and the exceſs of the ſquares of CB, BA is equal to the rectangle HG, GL.

Becauſe as CB to BD, ſo is OG to GL, the ſquare of CB
a. 1. 6. is to the rectangle CB, BD as [a] the rectangle HG, GO to the rectangle HG, GL: And the ſquare of CB is equal to the rectangle HG, GO, becauſe GO, BC, GH are proportionals;
b. 14. 5. therefore the rectangle CB, BD is equal [b] to HG, GL. And becauſe as CB to BD, ſo is OG to GL; twice CB is to BD as twice OG, that is, GN, to GL; and, by diviſion, as BC together with CD is to BD, ſo is NL, that is, LM, to LG;
c. 22. 6. Therefore [c] the ſquare of BC together with CD is to the ſquare of BD, as the ſquare of ML to the ſquare of LG: But the
d. 8 2. ſquare of BC and CD together is equal [d] to four times the rectangle BC, CD together with the ſquare of BD; therefore four times the rectangle BC, CD together with the ſquare of BD is to the ſquare of BD, as the ſquare of ML to the ſquare of LG: And, by diviſion, four times the rectangle BC, CD is to the ſquare of BD, as the ſquare of MG to the ſquare of GL; wherefore the rectangle BC, CD is to the ſquare of BD, as (the ſquare of KG the half of MG to the ſquare of GL, that is, as) the ſquare of AB to the ſquare of BD, becauſe as LG to GK, ſo DB was made to BA; Therefore [b] the rectangle BC, CD is equal to the ſquare of AB. To each of theſe add the rectangle CB, BD, and the ſquare of BC becomes equal to the ſquare of AB together with the rectangle CB, BD; therefore this rectangle, that is, the given rectangle HG, GL is the exceſs of the ſquares of BC, AB. From the point A, draw AP perpendicular to BC, and becauſe the angle ABP is equal to the angle EFG, the triangle ABP is equiangular to EFG: And DB was made to BA, as LG to GK; therefore as the rectangle CB, BD to CB, BA, ſo is the rectangle HG,



GL to HG, GK; and as the rectangle CB, BA to AP, BC, ſo is (the ſtraight line BA to AP, and ſo is FE or GK to EG,

G, and ſo is) the rectangle HG, GK to HG, GE; therefore, aequali, as the rectangle CB, BD to AP, BC, ſo is the ctangle HG, GL to EG, GH: And the rectangle CB, BD equal to HG, GL; therefore the rectangle AP, BC, that the parallelogram AC, is equal to the given rectangle EG, H.

PROP. LXXXVIII. N.

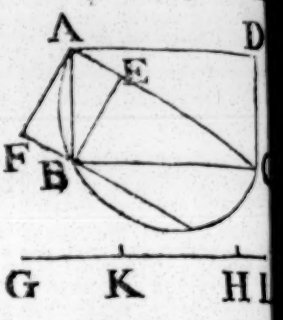
F two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the ſum of the ſquares its ſides be given, the ſides ſhall each of them be given.

Let the two ſtraight lines AB, BC contain the parallelogram BCD given in magnitude in the given angle ABC, and let e ſum of the ſquares of AB, BC be given; AB, BC are each them given.

Firſt, let ABC be a right angle; and becauſe twice the ctangle contained by two equal ſtraight lines is equal to both eir ſquares; but if two ſtraight lines are unual, twice the rectangle contained by them is s than the ſum of their ſquares, as is evident om the 7. Prop. B. 2. Elem. therefore twice e given ſpace, to which ſpace the rectangle of which the es are to be found is equal, muſt not be greater than the ven ſum of the ſquares of the ſides: And, if twice that ſpace equal to the given ſum of the ſquares, the ſides of the rectgle muſt neceſſarily be equal to one another: Therefore this caſe deſcribe a ſquare ABCD equal to the given rectgle, and its ſides AB, BC are thoſe which were to be und: For the rectangle AC is equal to the given ſpace, d the ſum of the ſquares of its ſides AB, BC is equal to twice e rectangle AC, that is, by the hypotheſis, to the given ſpace which the ſum of the ſquares was required to be equal.

But if twice the given rectangle be not equal to the given m of the ſquares of the ſides, it muſt be leſs than it, as s been ſhewn. Let ABCD be the rectangle, join AC and aw BE perpendicular to it, and complete the rectangle EBF, and deſcribe the circle ABC about the triangle ABC; C is its diameter [a]: And becauſe the triangle ABC is ſimi- [a] Cor. 5. 4
r [b] to AEB, as AC to CB, ſo is AB to BE; therefore the [b] 8. 6.
ctangle AC, BE is equal to AB, BC; and the rectangle AB,

BC is given, wherefore AC, BE is given: And becauſe the ſum of the ſquares of AB, BC is given, the ſquare of AC which
c. 47. 1. equal [c] to that ſum, is given; and AC itſelf is therefore give in magnitude: Let AC be likewiſe given in poſition, and th
d 32. Dat. point A; therefore AF is given [d] in poſition: And the rectangle AC, BE is given, as has been ſhewn, and AC is
e 61. Dat. given, wherefore [e] BE is given in magnitude, as alſo AF which is equal to it; and AF is alſo given in poſition, and
f 30. Dat. the point A is given; wherefore [f] the point F is given, and the ſtraight line
g 31. Dat. FB in poſition [g]: And the circumference
h 28. Dat. ABC is given in poſition, wherefore [h] the point B is given
i. 29. Dat. And the points A, C are given; therefore the ſtraight line AB, BC are given [i] in poſition and magnitude.

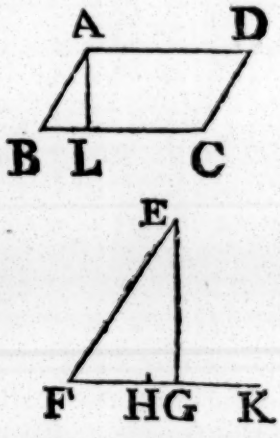


The ſides AB, BC of the rectangle may be found thus; l the rectangle GH, GK be the given ſpace to which the rectangle AB, BC is equal; and let GH, GL be the given rect angle to which the ſum of the ſquares of AB, BC is equal
k. 14. 2. Find [k] a ſquare equal to the rectangle GH, GL: And let i ſide AC be given in poſition; upon AC as a diameter deſcrib the ſemicircle ABC, and as AC to GH, ſo make GK to AF and from the point A place AF at right angles to AC: There
l. 16. 6. fore the rectangle CA, AF is equal [l] to GH, GK; and, b the hypotheſis, twice the rectangle GH, GK is leſs than GH GL, that is, than the ſquare of AC; wherefore twice th rectangle CA, AF is leſs than the ſquare of AC, and th rectangle CA, AF itſelf leſs than half the ſquare of AC, tha is, than the rectangle contained by the diameter AC and its half wherefore AF is leſs than the ſemidiameter of the circle, an conſequently the ſtraight line drawn thro' the point F paralle to AC muſt meet the circumference in two points: Let B b either of them, and join AB, BC, and complete the rectangl ABCD; ABCD is the rectangle which was to be found: Dra
m. 34. 1. BE perpendicular to AC; therefore BE is equal [m] to AF and becauſe the angle ABC in a ſemicircle is a right angle
b. 8. 6. the rectangle AB, BC is equal [b] to AC, BE, that is, to the rect angle CA, AF, which is equal to the given rectangle GH GK: And the ſquares of AB, BC are together equal [c] to th ſquare of AC, that is, to the given rectangle GH, GL.

Bu

But if the given angle ABC of the parallelogram AC be not a right angle, in this caſe, becauſe ABC is a given angle, the ratio of the rectangle contained by the ſides AB, BC to the parallelogram AC is given [n]; and AC is given, therefore the rectangle AB, BC is given; and the ſum of the ſquares of AB, BC is given; therefore the ſides AB, BC are given by the preceeding caſe. n 62. Dat.

The ſides AB, BC and the parallelogram AC may be found thus: Let EFG be the given angle of the parallelogram, and from any point E in FE draw EG perpendicular to FG; and let the rectangle EG, FH be the given ſpace to which the parallelogram is to be made equal, and let EF, FK be the given rectangle to which the ſum of the ſquares of the ſides is to be equal. And, by the preceeding caſe, find the ſides of a rectangle which is equal to the given rectangle EF, FH, and the ſquares of the ſides of which are together equal to the given rectangle EF, FK; therefore, as was ſhewn in that caſe, twice the rectangle EF, FH muſt not be greater than the rectangle EF, FK; let it be ſo, and let AB, BC be the ſides of the rectangle joined in the angle ABC equal to the given angle EFG; and complete the parallelogram ABCD, which will be that which was to be found: Draw AL perpendicular to BC, and becauſe the angle ABL is equal to EFG, the triangle ABL is equiangular to EFG; and the parallelogram AC, that is, the rectangle AL, BC, is to the rectangle AB, BC as (the ſtraight line AL to AB, that is, as EG to EF, that is, as) the rectangle EG, FH to EF, FH; and, by the conſtruction, the rectangle AB, BC is equal to EF, FH, therefore the rectangle AL, BC, or its equal, the parallelogram AC, is equal to the given rectangle EG, FH; and the ſquares of AB, BC are together equal, by conſtruction, to the given rectangle EF, FK.



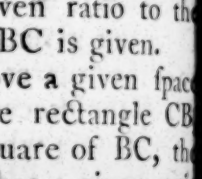
PROP.

80. PROP. LXXXIX.

IF two ſtraight lines contain a given parallelogram in a given angle, and if the exceſs of the ſquare of one of them above a given ſpace has a given ratio to the ſquare of the other; each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the exceſs of the ſquare of BC above a given ſpace have a given ratio to the ſquare of AB; each of the ſtraight lines AB, BC is given.

Becauſe the exceſs of the ſquare of BC above a given ſpace has a given ratio to the ſquare of BA, let the rectangle CB, BD be the given ſpace; take this from the ſquare of BC, the
a. 2. 2. remainder, to wit, the rectangle [a] BC, CD has a given ratio to the ſquare of BA: Draw AE perpendicular to BC, and let the ſquare of BF be equal to the rectangle BC, CD; then becauſe the angle ABC, as alſo BEA, is given,
b 43. Dat. the triangle ABE is given [b] in ſpecies, and the ratio of AE to AB given: And becauſe the ratio of the rectangle BC, CD, that is, of the ſquare of BF to the ſquare of BA, is given, the ratio of the ſtraight line BF to BA



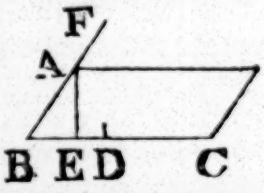
c 58. Dat. is given [c]; and the ratio of AE to AB is given, wherefore [d] the
d. 9. Dat. ratio of AE to BF is given; as alſo the ratio of the rectangle
e 35. 1. AE, BC, that is, [e] of the parallelogram AC to the rectangle FB, BC; and AC is given, wherefore the rectangle FB, BC is given. The exceſs of the ſquare of BC above the ſquare of BF, that is, above the rectangle BC, CD, is given, for it is equal [a] to the given rectangle CB, BD; therefore, becauſe the rectangle contained by the ſtraight lines FB, BC is given, and alſo the
f 87. Dat. exceſs of the ſquare of BC above the ſquare of BF; FB, BC are each of them given [f]; and the ratio of FB to BA is given, therefore, AB BC are given.

The Compoſition is as follows.

Let GHK be the given angle to which the angle of the parallelogram is to be made equal, and from any point G in HG, draw GK perpendicular to HK; let GK, HL be the rect-
angle

gle to which the parallelogram is to be
de equal, and let LH, HM be the rect-
gle equal to the given ſpace which is to
taken from the ſquare of one of the
es; and let the ratio of the remainder to
ſquare of the other ſide be the ſame with
ratio of the ſquare of the given ſtraight line NH to the
are of the given ſtraight line HG.

By help of the 87. Dat. find two ſtraight lines BC, BF
ich contain a rectangle equal to the given rectangle NH,
, and ſuch that the exceſs of the ſquare
BC above the ſquare of BF be equal to
given rectangle LH, HM; and join CB
in the angle FBC equal to the given
gle GHK: And as NH to HG, ſo make
to BA, and complete the parallelogram
, and draw AE perpendicular to BC; then AC is equal to
rectangle GK, HL; and if from the ſquare of BC, the gi-
rectangle LH, HM be taken, the remainder ſhall have to
ſquare of BA the ſame ratio which the ſquare of NH has to
ſquare of HG.



Becauſe, by the conſtruction, the ſquare of BC is equal to
ſquare of BF together with the rectangle LH, HM; if
m the ſquare of BC there be taken the rectangle LH, HM,
e remains the ſquare of BF which has [g] to the ſquare of g 22. 6.
the ſame ratio which the ſquare of NH has to the ſquare
HG, becauſe, as NH to HG, ſo FB was made to BA; but
HG to GK, ſo is BA to AE, becauſe the triangle GHK is
iangular to ABE; therefore, ex æquali, as NH to GK,
s FB to AE; wherefore [h] the rectangle NH, HL is to the h 1. 6.
angle GK, HL, as the rectangle FB, BC to AE, BC; but,
he conſtruction, the rectangle NH, HL is equal to FB,
; therefore [k] the rectangle GK, HL is equal to the rect- k 14. 5.
le AE, BC, that is, to the parallelogram AC.

The analyſis of this Problem might have been made as in the
Prop. in the Greek, and the compoſition of it may be made
hat which is in Prop. 87. of this edition.

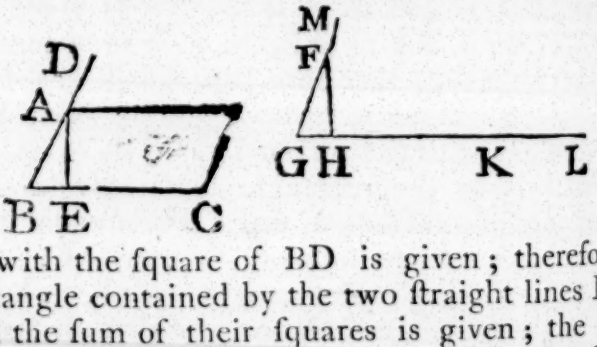
PROP

O. PROP. XC.

IF two ſtraight lines contain a given parallelogram in a given angle, and if the ſquare of one of them together with the ſpace which has a given ratio to the ſquare of the other be given; each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the ſquare of B together with the ſpace which has a given ratio to the ſquare of AB be given; AB, BC are each of them given.

Let the ſquare of BD be the ſpace which has the given ratio to the ſquare of AB; therefore, by the hypotheſis, the ſquare of BC together with the ſquare of BD is given. From the point A, draw AE perpendicular to BC; and becauſe the angles
a 43. Dat. ABE, BEA are given, the triangle ABE is given [a] in ſpecies; therefore the ratio of BA to AE is given: And becauſe the ratio of the ſquare of BD to the ſquare of BA is given, the ratio
b 58. Dat. of the ſtraight line BD to BA is given [b]; and the ratio
c 9. Dat. BA to AE is given, therefore, [c] the ratio of AE to BD is given, as alſo the ratio of the rectangle AE, BC, that is, of the parallelogram AC to the rectangle DB, BC; and AC is given, therefore the rectangle DB, BC is given; and the ſquare



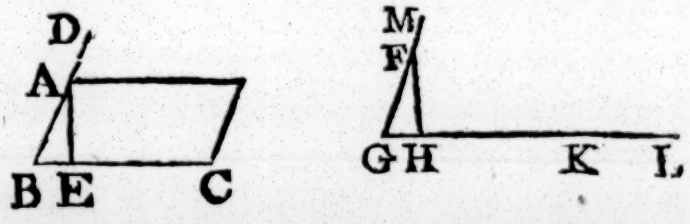
BC together with the ſquare of BD is given; therefore [d] becauſe the rectangle contained by the two ſtraight lines DB,
d 88. Dat. is given, and the ſum of their ſquares is given; the ſtraight lines DB, BC are each of them given; and the ratio of DB to BA is given; therefore AB, BC are given.

The Compoſition is as follows.

Let FGH be the given angle to which the angle of the parallelogram is to be made equal, and from any point F in draw FH perpendicular to GH; and let the rectangle GK be that to which the parallelogram is to be made equal; let the rectangle KG, GL be the ſpace to which the ſq

of one of the ſides of the parallelogram together with the ſpace which has a given ratio to the ſquare of the other ſide, is to be made equal; and let this given ratio be the ſame which the ſquare of the given ſtraight line MG has to the ſquare of GF.

By the 88. Dat. find two ſtraight lines DB, BC which contain a rectangle equal to the given rectangle MG, GK, and ſuch that the ſum of their ſquares is equal to the given rectangle KG, GL; therefore, by the determination of the Problem in that Propoſition, twice the rectangle MG, GK muſt not be greater than the rectangle KG, GL. Let it be ſo, and join the ſtraight lines DB, BC in the angle DBC equal to the given angle FGH; and as MG to GF, ſo make DB to BA, and complete the parallelogram AC: AC is equal to the rect-



angle FH, GK; and the ſquare of BC together with the ſquare of BD, which, by the conſtruction, has to the ſquare of BA the given ratio which the ſquare of MG has to the ſquare of GF, is equal, by the conſtruction, to the given rectangle KG, GL. Draw AE perpendicular to BC.

Becauſe as DB to BA, ſo is MG to GF; and as BA to AE, ſo GF to FH; ex æquali, as DB to AE, ſo is MG to FH; therefore as the rectangle DB, BC to AE, BC, ſo is the rectangle MG, GK to FH, GK; and the rectangle DB, BC is equal to the rectangle MG, GK; therefore the rectangle AE, BC, that is, the parallelogram AC, is equal to the rectangle FH, GK.

PROP. XCI. 88.

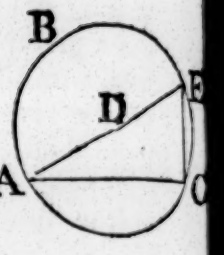
IF a ſtraight line drawn within a circle given in magnitude cuts off a ſegment which contains a given angle; the ſtraight line is given in magnitude.

In the circle ABC given in magnitude, let the ſtraight line AC be drawn, cutting off the ſegment AEC which contains the given angle AEC; the ſtraight line AC is given in magnitude.

Take D the center of the circle [a], join AD and produce it [a] 1. 3.

to

to E, and join EC: The angle ACE being
b 31. 3. a right [b] angle is given; and the angle
c 43. Dat. AEC is given; therefore [c] the triangle ACE is given in ſpecies, and the ratio of EA to AC is therefore given; and EA is
d 5. Def. given [d] in magnitude, becauſe the circle is given in magnitude; AC is therefore gi-
e 2. Dat. ven [e] in magnitude.



89. PROP. XCII.

IF a ſtraight line given in magnitude be drawn within a circle given in magnitude; it ſhall cut off a ſegment containing a given angle.

Let the ſtraight line AC given in magnitude be drawn within the circle ABC given in magnitude; it ſhall cut off a ſegment containing a given angle.

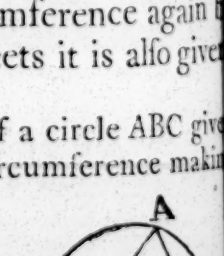
Take D the center of the circle, join AD and produce it to E, and join EC: And becauſe each of the ſtraight lines EA,
a 1. Dat. and AC is given, their ratio is given [a]; and the angle ACE is a right angle, therefore
b 46. Dat. the triangle ACE is given [b] in ſpecies, and conſequently, the angle AEC is given.

90. PROP. XCIII.

IF from any point in the circumference of a circle given in poſition two ſtraight lines be drawn meeting the circumference and containing a given angle; if the point in which one of them meets the circumference again be given, the point in which the other meets it is alſo given.

From any point A in the circumference of a circle ABC given in poſition, let AB, AC be drawn to the circumference making the given angle BAC; if the point B be given, the point C is alſo given.

Take D the center of the circle, and join BD, DC; and becauſe each of the
a 29. Dat. points B, D is given, BD is given [a] in poſition; and becauſe the angle BAC is gi-
b 20. 3. ven, the angle BDC is given [b], therefore



becau

because the ſtraight line DC is drawn to the given point D in the ſtraight line BD given in poſition in the given angle BDC, DC is given [c] in poſition: And the circumference ABC is given in poſition, therefore [d] the point C is given.

c 32. Dat.
d 28. Dat.

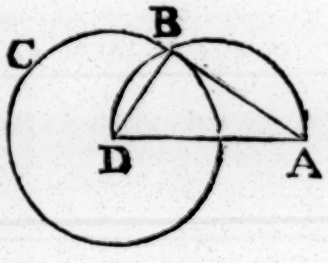
PROP. XCIV.

91.

IF from a given point a ſtraight line be drawn touching a circle given in poſition; the ſtraight line is given in poſition and magnitude.

Let the ſtraight line AB be drawn from the given point A touching the circle BC given in poſition; AB is given in poſition and magnitude.

Take D the center of the circle, and join DA, DB: Becauſe each of the points D, A is given, the ſtraight line AD is given [a] in poſition and magnitude: And DBA is a right [b] angle, wherefore DA is a diameter [c] of the circle DBA, deſcribed about the triangle DBA; and that circle is therefore given [d] in poſition: And the circle BC is given in poſition, therefore the point B is given [e]: The point A is alſo given; therefore the ſtraight line AB is given [a] in poſition and magnitude.



a 29. Dat.
b. 18. 3.
c. Cor. 5. 4.
d. 6 Def
e 28. Dat.

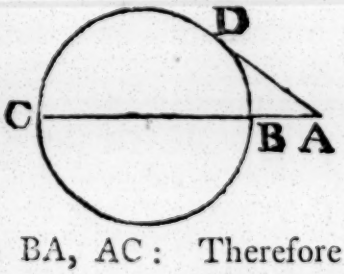
PROP. XCV.

92.

IF a ſtraight line be drawn from a given point without a circle given in poſition; the rectangle contained by the ſegments betwixt the point and the circumference of the circle is given.

Let the ſtraight line ABC be drawn from the given point A without the circle BCD given in poſition, cutting it in B, C; the rectangle BA, AC is given.

From the point A, draw [a] AD touching the circle; therefore AD is given in poſition and magnitude: And becauſe AD is given, the ſquare of AD is given [c] which is equal [d] to the rectangle BA, AC: Therefore the rectangle BA, AC is given.



a. 17. 3.
b 94. Dat.
c 56. Dat.
d 36. 3.

PROP.

93. PROP. XCVI.

IF a ſtraight line be drawn thro' a given point within a circle given in poſition, the rectangle contained by the ſegments betwixt the point and the circumference of the circle is given.

Let the ſtraight line BAC be drawn thro' the given point A within the circle BCE given in poſition; the rectangle BA, AC is given.

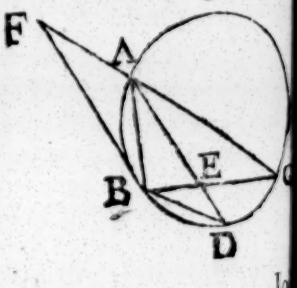
Take D the center of the circle, join AD and produce it to the points E, F. Becauſe the points A, D are given, the ſtraight line AD is given [a] in poſition; and the circle BEC is given in poſition; therefore the points E, F are given [b], and the point A is given, therefore EA, AF are each of them given [a]; and the rectangle EA, AF is therefore given; and it is equal [c] to the rectangle BA, AC, which conſequently is given.

a. 29. Dat.
b. 28. Dat.
c. 35. 3.

94. PROP. XCVII.

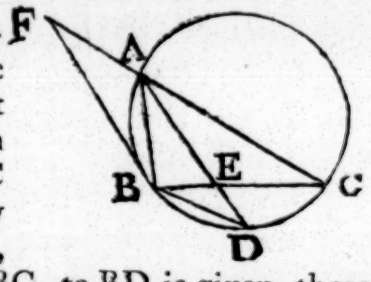
IF a ſtraight line be drawn within a circle given in magnitude cutting off a ſegment containing a given angle; if the angle in the ſegment be biſected by a ſtraight line produced till it meets the circumference, the ſtraight lines which contain the given angle ſhall both of them together have a given ratio to the ſtraight line which biſects the angle, and the rectangle contained by both theſe lines together which contain the given angle and the part of the biſecting line cut off below the baſe of the ſegment, ſhall be given.

Let the ſtraight line BC be drawn within the circle AB given in magnitude cutting off a ſegment containing the given angle BAC, and let the angle BAC be biſected by the ſtraight line AD; BA together with AC has a given ratio to AD; and the rectangle contained by BA and AC together, and the ſtraight line ED cut off from AD below BC the baſe of the ſegment, is given.



Join BD; and becauſe BC is drawn within the circle ABC given in magnitude cutting off the ſegment BAC containing the given angle BAC; BC is given [a] in magnitude: By the ſame reaſon BD is given; therefore [b] the ratio of BC to BD is given: And becauſe the angle BAC is biſected by AD, as BA to AC, ſo is [c] BE to EC; and, by permutation, as AB to BE, ſo is AC to CE; wherefore [d] as BA and AC together to BC, ſo is AC to CE: And becauſe the angle BAE is equal to EAC, and the angle ACE to [e] ADB; the triangle ACE is equiangular to the triangle ADB; therefore as AC to CE, ſo is AD to DB: But as AC to CE, ſo is BA together with AC to BC; as therefore BA and AC to BC, ſo is AD to DB; and, by permutation, as BA and AC to AD, ſo is BC to BD: And the ratio of BC to BD is given, therefore the ratio of BA together with AC to AD is given.

a 91. Dat.
b. 1. Dat.
c. 3. 6.
d. 12. 5.
e. 21. 3.



Alſo the rectangle contained by BA and AC together, and DE is given.

Becauſe the triangle BDE is equiangular to the triangle ACE, as BD to DE, ſo is AC to CE; and as AC to CE, ſo is BA and AC to BC; therefore as BA and AC to BC, ſo is BD to DE; wherefore the rectangle contained by BA and AC together, and DE is equal to the rectangle CB, BD: But CB, BD is given; therefore the rectangle contained by BA and AC together, and DE, is given.

Otherwiſe.

Produce CA, and make AF equal to AB, and join BF: and becauſe the angle BAC is double [a] of each of the angles BFA, BAD, the angle BFC is equal to BAD; and the angle BCA is equal to BDA, therefore the triangle FCB is equiangular to ADB: As therefore FC to CB, ſo is AD to DB; and, by permutation, as FC, that is, BA and AC together to AD, ſo is CB to BD: And the ratio of CB to BD is given, therefore the ratio of BA and AC to AD is given.

a { 5. & 32. 1.

And becauſe the angle BFC is equal to the angle DAC, that is, to the angle DBC, and the angle ACB equal to the angle ADB; the triangle FCB is equiangular to BDE, as therefore FC to CB, ſo is BD to DE; therefore the rectangle contained by FC, that is, BA and AC together, and DE is equal

qual to the rectangle CB, BD which is given, and therefo the rectangle contained by BA, AC together, and DE is give

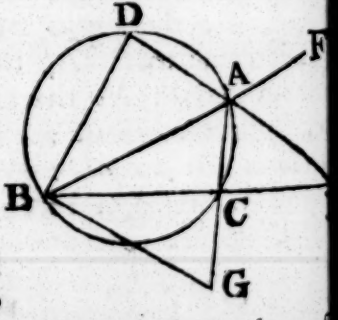
P. PROP. XCVIII.

IF a ſtraight line be drawn within a circle given magnitude, cutting off a ſegment containing a giv angle; if the angle adjacent to the angle in the ſegme be biſected by a ſtraight line produced till it meet t circumference again and the baſe of the ſegment; t exceſs of the ſtraight lines which contain the giv angle ſhall have a given ratio to the ſegment of the b ſecting line which is within the circle; and the rectang contained by the ſame exceſs and the ſegment of the b ſecting line betwixt the baſe produced and the poi where it again meets the circumference, ſhall be give

Let the ſtraight line BC be drawn within the circle AE given in magnitude cutting off a ſegment containing the giv angle BAC, and let the angle CAF adjacent to BAC be ſected by the ſtraight line DAE meeting the circumferen again in D, and BC the baſe of the ſegment produced in the exceſs of BA, AC has a given ratio to AD; and the re angle which is contained by the ſame exceſs and the ſtraig line ED, is given.

Join BD, and thro' B, draw BG parallel to DE meeti AC produced in G: And becauſe BC cuts off from the cir ABC given in magnitude the ſegment BAC containing a given angle, BC is therefore given[a] in magnitude: By the ſame reaſon BD is given, becauſe the angle BAD is equal to the given angle EAF; therefore the ratio of BC to BD is given: And becauſe the angle CAE is equal to EAF, of which CAE is equal to the alternate angle AGB, and EAF to the interior and oppo angle ABG; therefore the angle AGB is equal to ABG, a the ſtraight line AB equal to AG; ſo that GC is the exc of BA, AC: And becauſe the angle BGC is equal to GA that is, to EAF, or the angle BAD; and that the angle BC

a.91. Dat.



s equal to the oppoſite interior angle BDA of the quadrilateral CAD in the circle; therefore the triangle BGC is equiangu- r to BDA: therefore as GC to CB, ſo is AD to DB; and, y permutation, as GC, which is the exceſs of BA, AC to AD, ſo is CB to BD: And the ratio of CB to BD is given; herefore the ratio of the exceſs of BA, AC to AD is given.

And becauſe the angle GBC is equal to the alternate angle DEB, and the angle BCG equal to BDE; the triangle BCG s equiangular to BDE: Therefore as GC to CB, ſo is BD to DE; and conſequently the rectangle GC, DE is equal to the rectangle CB, BD which is given, becauſe its ſides CB, BD re given: Therefore the rectangle contained by the exceſs of BA, AC and the ſtraight line DE is given.

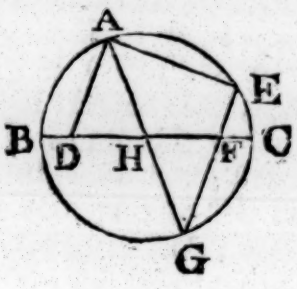
PROP. XCIX. 95.

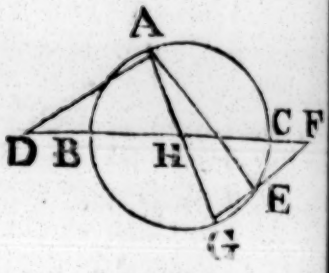
IF from a given point in the diameter of a circle given in poſition, or in the diameter produced, a ſtraight ne be drawn to any point in the circumference, and from hat point a ſtraight line be drawn at right angles to the irſt, and from the point in which this meets the cir- umference again, a ſtraight line be drawn parallel to he firſt; the point in which this parallel meets the di- meter is given; and the rectangle contained by the wo parallels is given.

In BC the diameter of the circle ABC given in poſition, or BC produced, let the given point D be taken, and from D a ſtraight line DA be drawn to any point A in the cir- umference, and let AE be drawn at right angles to DA, and om the point E where it meets the circumference again let F be drawn parallel to DA meeting BC in F; the point F given, as alſo the rectangle AD, EF.

Produce EF to the circumference in G, and join AG: Be- uſe GEA is a right angle, the ſtraight line AG is [a] the di- meter of the circle ABC; and BC is alſo a diameter of it; herefore the point H where they meet is the center of the rcle, and conſequently H is given: And the point D is given, herefore DH is given in magnitude: And becauſe AD is pa- [a] Cor. 5. 4.

b. 4. 6. rallel to FG, and GH equal to HA; DH is equal [b] to F and AD equal to GF: And DH is given, therefore HF is





c. 30. Dat. ven in magnitude, ; and it is alſo given in poſition, and point H is given, therefore [c] the point F is given.

And becauſe the ſtraight line EFG is drawn from a gi point F without or within the circle ABC given in poſit d 95. or 96. Dat. therefore [d] the rectangle EF, FG is given: And GF is e to AD, wherefore the rectangle AD, EF is given.

Q.

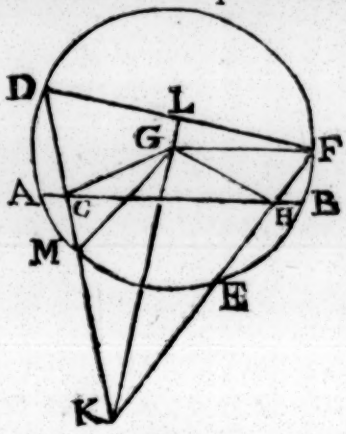
PROP. C.

IF from a given point in a ſtraight line given in ſition, a ſtraight line be drawn to any point in the cumference of a circle given in poſition; and from point a ſtraight line be drawn making with the firſt angle equal to the difference of a right angle and angle contained by the ſtraight line given in poſition, the ſtraight line which joins the given point and the ce of the circle; and from the point in which the ſec line meets the circumference again, a third ſtraight be drawn making with the ſecond an angle equal to which the firſt makes with the ſecond: The point in wh this third line meets the ſtraight line given in poſitio given; as alſo the rectangle contained by the 1ſt ſtra line and the ſegment of the third betwixt the circu rence and the ſtraight line given in poſition, is given

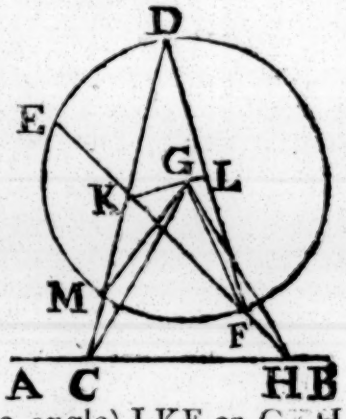
Let the ſtraight line CD be drawn from the given poin in the ſtraight line AB given in poſition, to the circumfer of the circle DEF given in poſition of which G is the ce join CG, and from the point D let DF be drawn making angle CDF equal to the difference of a right angle and angle BCG, and from the point F let FE be drawn ma

angle DFE equal to CDF, meeting AB in H : The point H is ven; as alſo the rectangle CD, FH.

Let CD, FH meet one another in point K, from which draw KL rpendicular to DF; and let DC ect the circumference again in M, d let FH meet the ſame in E, and n MG, GF, GH.



Becauſe the angles MDF, DFE are ual to one another, the circumfe- nces MF, DE are equal [a]; and add- g or taking away the common part E, the circumference DM is equal EF; therefore the ſtraight line DM equal to the ſtraight line EF, and e angle GMD to the angle [b] GFE; d the angles GMC, GFH are equal one another, becauſe they are ei- er the ſame with the angles GMD, FE, or adjacent to them: And be- uſe the angles KDL, LKD are toge- er equal [c] to a right angle, that is, the hypotheſis, to the angles KDL, CB; the angle GCB or GCH is e- ual to the angle (LKD, that is, to the angle) LKF or GKH: herefore the points C, K, H, G are in the circumference a circle; and the angle GCK is therefore equal to the an- e GHF; and the angle GMC is equal to GFH, and the raight line GM to GF; therefore [d] CG is equal to GH, d CM to HF: And becauſe CG is equal to GH, the ngle GCH is equal to GHC; but the angle GCH is given: herefore GHC is given, and conſequently the angle CGH given; and CG is given in poſition, and the point ; therefore [e] GH is given in poſition; and CB is alſo given poſition, wherefore the point H is given.



a. 26. 3.

b. 8. 1.

c. 32. 1.

d. 26. 1.

e 32. Dat.

And becauſe HF is equal to CM, the rectangle DC, FH is qual to DC, CM: But DC, CM is given [f], becauſe the oint C is given; therefore the rectangle DC, FH is given.

f 95. or 96 Dat.

FINIS.

NOTES

ON

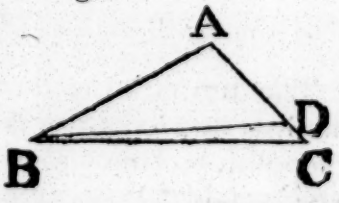
EUCLID'S DATA.

DEFINITION II.

THIS is made more explicit than in the Greek text, to prevent a miſtake which the Author of the ſecond demonſtration of the 24th Propoſition in the Greek Edition has fallen into, of thinking that a ratio is given to which another ratio is ſhewn to be equal, tho' this other be not exhibited in given magnitudes. See the Notes on that Propoſition, which is the 13th in this edition. Beſides, by this definition, as it is now given, ſome Propoſitions are demonſtrated, which in the Greek are not ſo well done by help of Prop. 2.

DEF. IV.

In the Greek text, Def. 4. is thus: "Points, lines, ſpaces, and angles are ſaid to be given in poſition which have always the ſame ſituation;" but this is imperfect and uſeleſs, becauſe there are innumerable caſes in which things may be given according to this definition, and yet their poſition cannot be found; for inſtance, let the triangle ABC be given in poſition, and let it be propoſed to draw a ſtraight line BD from the angle at B to the oppoſite ſide AC which ſhall cut off the angle DBC which ſhall be the ſeventh part of the angle ABC, ſuppoſe this is done, therefore the ſtraight line BD is invariable in its poſition, that is, has always the ſame ſituation; for any other ſtraight line drawn from the point B on either ſide of BD cuts off an angle greater or leſſer than the ſeventh part of the angle ABC; therefore, according to this definition, the ſtraight line BD is given in poſition, as alſo[a] the point D in which it meets the ſtraight line AC which is given in poſition. But from the things here given, neither the ſtraight line BD nor the point D can be found by the help of Euclid's Elements only, by which every thing in his Data is



[a] 28. Dat.

ſuppoſed may be found. This Definition is therefore of uſe. We have amended it by adding "and which are cith "actually exhibited or can be found;" for nothing is to reckoned given, which cannot be found, or is not actually e hibited.

The Definition of an angle given by poſition is taken out the 4th, and given more diſtinctly by itſelf in the Definiti marked A.

DEF. XI. XII. XIII. XIV. XV.

The 11th and 12th are omitted, becauſe they cannot be g ven in Engliſh ſo as to have any tolerable ſenſe; and therefor wherever the terms defined occur, the words which expr their meaning are made uſe of in their place.

The 13. 14. 15. are omitted, as being of no uſe.

It is to be obſerved in general of the Data in this boo that they are to be underſtood to be given Geometrically, n always Arithmetically, that is, they cannot always be exhib ed in numbers; for inſtance, if the ſide of a ſquare be g b. 44. Dat. ven, the ratio of it to its diameter is given [b] geometricall c. 2. Dat. but not in numbers; and the diameter is given [c]; but th the number of any equal parts in the ſide be given, for e ample 10, the number of them in the diameter cannot be g ven: And the like holds in many other caſes.

PROPOSITION I.

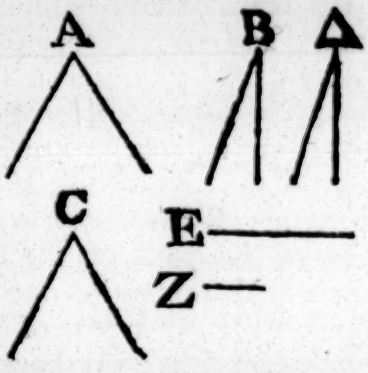
In this it is ſhewn that A is to B, as C to D, from thi that A is to C, as B to D, and then by permutation; b it follows directly, without theſe two ſteps, from 7. 5.

PROP. II.

The limitation added at the end of this Propoſition betwe the inverted commas is quite neceſſary, becauſe without it t Propoſition cannot always be demonſtrated: For the Auth having ſaid * "becauſe A is given, a magnitude equal to a. 1. Def. "can be found [a]; let this be C; and becauſe the ratio of b. 2. Def. "to B is given, a ratio which is the ſame to it can be found [b] adds, "let it be found, and let it be the ratio of C to Δ Now, from the ſecond Definition nothing more follow than that ſome ratio, ſuppoſe the ratio of E to Z, can b found which is the ſame with the ratio of A to B; an when the Author ſuppoſes that the ratio of C to Δ, which al

* See Dr Gregory's Edition of the Data.

lſo the ſame with the ratio of A to B, can be found, he neceſſarily ſuppoſes that to the three magnitudes E, Z, C a fourth proportional △ may be found; but this cannot always be done by the Elements of Euclid; from which it is plain Euclid muſt have underſtood the Propoſition under the limitation which is now added to his text. An example will make this clear; let A be a given angle, and B another angle to which A has a given ratio, for inſtance, the ratio of the given ſtraight line E to the given one Z; then, having found an angle C equal to A, how can the angle △ be found to which C has the ſame ratio that E has to Z? certainly no way, until it be ſhewn how to find an angle to which a given angle has a given ratio, which cannot be done by Euclid's Elements, nor probably by any Geometry known in his time. Therefore, in all the Propoſitions of this book which depend upon this ſecond, the above mentioned limitation muſt be underſtood, though it be not explicitly mentioned.



PROP. V.

The order of the Propoſitions in the Greek text between Prop. 4. and Prop. 25. is now changed into another which is more natural, by placing thoſe which are more ſimple before thoſe which are more complex; and by placing together thoſe which are of the ſame kind, ſome of which were mixed among others of a different kind. Thus, Prop. 12. in the Greek is now made the 5. and thoſe which were the 22. and 23. are made the 11. and 12. as they are more ſimple than the Propoſitions concerning magnitudes, the exceſs of one of which above a given magnitude has a given ratio to the other, after which theſe two were placed; and the 24. in the Greek text is, for the ſame reaſon, made the 13.

PROP. VI. VII.

These are univerſally true, tho' in the Greek text they are demonſtrated by Prop. 2. which has a limitation; they are therefore now ſhewn without it.

PROP.

PROP. XII.

In the 23. Prop. in the Greek text, which here is the 1 the words, " μη τὰς αὐτὰς δέ," are wrong tranſlated b Claud. Hardy, in his Edition of Euclid's Data, printed at Pa ris, Ann. 1625, which was the firſt edition of the Gree text; and Dr Gregory follows him in tranſlating them by th words, " etſi non eaſdem," as if the Greek had bee καὶ μὴ ἴδας αὐτάς, as in Prop. 9. of the Greek text. Euclid meaning is, that the ratios mentioned in the Propoſition mu not be the ſame; for, if they were, the Propoſition would no be true: Whatever ratio the whole has to the whole, if the ra tios of the parts of the firſt to the parts of the other be th ſame with this ratio, one part of the firſt may be double, triple &c. of the other part of it, or have any other ratio to it, an conſequently cannot have a given ratio to it; wherefore, thes words muſt be rendered by " non autem eaſdem," but not th ſame ratios, as Zambertus has tranſlated them in his Edition

PROP. XIII.

Some very ignorant Editor has given a ſecond demonſtra tion of this Propoſition in the Greek text, which has been a ignorantly kept in it by Claud. Hardy and Dr Gregory, an has been retained in the tranſlations of Zambertus and others Carolus Renaldinus gives it only: The author of it has though that a ratio was given if another ratio could be ſhewn to b the ſame to it, though this laſt ratio be not found: But this i altogether abſurd, becauſe from it would be deduced that th ratio of the ſides of any two ſquares is given, and the ratio o the diameters of any two circles, &c. And it is to be obſerved that the moderns frequently take given ratios, and ratios tha are always the ſame, for one and the ſame thing; and Sir Iſaa Newton has fallen into this miſtake in the 17th Lemma of hi Principia, Ed. 1713, and in other places; but this ſhould b carefully avoided, as it may lead into other errors.

PROP. XIV. XV.

Euclid in this book has ſeveral Propoſitions concernin magnitudes, the exceſs of one of which above a given mag nitud

itude has a given ratio to the other; but he has given none oncerning magnitudes whereof one together with a given agnitude has a given ratio to the other; tho' theſe laſt oc- ur as frequently in the ſolution of Problèms as the firſt; the eaſon of which is, that the laſt may be all demonſtrated by elp of the firſt; for, if a magnitude together with a given agnitude has a given ratio to another magnitude, the exceſs f this other above a given magnitude ſhall have a given ra- io to the firſt, and on the contrary; as we have demonſtrated n Prop. 14. And for a like reaſon Prop. 15. has been added o the Data. One example will make the thing clear; ſuppoſe were to be demonſtrated, that if a magnitude A together ith a given magnitude has a given ratio to another magni- ude B, that the two magnitudes A and B, together with a iven magnitude, have a given ratio to that other magnitude ; which is the ſame Propoſition with reſpect to the laſt kind f magnitudes above mentioned, that the firſt part of Prop. 6. in this Edition is in reſpect of the firſt kind: This is ſhewn hus; from the hypotheſis, and by the firſt part of Prop. 14. he exceſs of B above a given magnitude has unto A a given atio; and therefore, by the firſt part of Prop. 17. the exceſs of above a given magnitude has unto B and A together a given atio; and by the ſecond part of Prop. 14. A and B together ith a given magnitude has unto B a given ratio; which is the hing that was to be demonſtrated. In like manner, the other ropoſitions concerning the laſt kind of magnitudes may be hewn.

PROP. XVI. XVII.

In the third part of Prop. 10. in the Greek text, which is he 16. in this Edition, after the ratio of EC to CB has been hewn to be given; from this, by inverſion and converſion, he ratio of BC to BE is demonſtrated to be given; but, with- ut theſe two ſteps, the concluſion ſhould have been made nly by citing the 6. Propoſition. And in like manner, in the irſt part of Prop. 11. in the Greek, which in this edition is he 17. from the ratio of DB to BC being given, the ratio of DC o DB is ſhewn to be given, by inverſion and compoſition, in- ſtead of citing Prop. 7. and the ſame fault occurs in the ſecond art of the ſame Prop. 11.

PROP.

PROP. XXI. XXII.

These are now added, as being wanting to complete the subject treated of in the four preceeding Propositions.

PROP. XXIII.

This which is Prop. 20. in the Greek text, was separated from Prop. 14. 15. 16. in that text, after which it should have been immediately placed, as being of the same kind; it is now put into its proper place; but Prop. 21. in the Greek is left out, as being the same with Prop. 14. in that text, which is here Prop. 18.

PROP. XXIV.

This, which is Prop. 13. in the Greek, is now put into its proper place, having been disjoined from the three following in this edition, which are of the same kind.

PROP. XXVIII.

This which in the Greek text is Prop. 25. and several of the following Propositions are there deduced from Def. 4. which is not sufficient, as has been mentioned in the note on that definition: They are therefore now shewn more explicitly.

PROP. XXXIV. XXXVI.

Each of these has a determination, which is now added, which occasions a change in their demonstrations.

PROP. XXXVII. XXXIX. XL. XLI.

The 35. and 36. Propositions in the Greek text are joined into one, which makes the 39. in this edition, because the same enunciation and demonstration serves both: And for the same reason Prop. 37. 38. in the Greek are joined into one, which here is the 40.

Prop. 37. is added to the Data, as it frequently occurs in the solution of Problems; and Prop. 41. is added to complete the rest.

PROP. XLII.

This is Prop. 39. in the Greek text, where the whole construction of Prop. 22. of Book I. of the Elements is put, without need, into the demonstration, but is now only cited.

PROP. XLV.

This is Prop. 42. in the Greek, where the three straight lines made use of in the construction are said, but not shewn to be such that any two of them is greater than the third, which is now done.

PRO

PROP. XLVII.

This is Prop. 44. in the Greek text; but the demonſtration of it is changed into another wherein the ſeveral caſes of it are ſhewn, which, tho' neceſſary, is not done in the Greek.

PROP. XLVIII.

There are two caſes in this Propoſition, ariſing from the two caſes of the third part of Prop. 47. on which the 48. depends; and in the compoſition theſe two caſes are explicitly given.

PROP. LII.

The conſtruction and demonſtration of this, which is Prop. 48. in the Greek, are made ſomething ſhorter than in that text.

PROP. LIII.

Prop. 63. in the Greek text is omitted, being only a caſe of Prop. 49. in that text, which is Prop. 53. in this edition.

PROP. LVIII.

This is not in the Greek text, but its demonſtration is contained in that of the firſt part of Prop. 54. in that text; which Propoſition is concerning figures that are given in ſpecies; this 58. is true of ſimilar figures, though they be not given in ſpecies, and, as it frequently occurs, it was neceſſary to add it.

PROP. LIX. LXI.

This is the 54. in the Greek; and the 77. in the Greek, being the very ſame with it, is left out, and a ſhorter demonſtration is given of Prop. 61.

PROP. LXII.

This, which is moſt frequently uſeful, is not in the Greek, and is neceſſary to Prop. 87. 88. in this edition, as alſo, though not mentioned, to Prop. 86. 87. in the former editions. Prop. 66. in the Greek text is made a corollary to it.

PROP. LXIV.

This contains both Prop. 74. and 73. in the Greek text; the firſt caſe of the 74. is a repetition of Prop. 56. from which it is ſeparated in that text by many Propoſitions; and as there is no order in theſe Propoſitions, as they ſtand in the Greek, they are now put into the order which ſeemed moſt convenient and natural.

The

The demonſtration of the firſt part of Prop. 73. in t Greek is groſsly vitiated. Dr Gregory ſays that the ſente ces he has incloſed betwixt two ſtars are ſuperfluous and oug to be cancelled; but he has not obſerved that what follo them is abſurd, being to prove that the ratio [See his figur of Aγ to γK is given, which by the hypotheſis at the begi ning of the Propoſition is expreſsly given; ſo that the whole this part was to be altered, which is done in this Prop. 64.

PROP. LXVII. LXVIII.

Prop. 70. in the Greek text is divided into theſe two, f the ſake of diſtinctneſs; and the demonſtration of the 67. rendered ſhorter than that of the firſt part of Prop. 70. the Greek, by means of Prop. 23. of Book 6. of the El ments.

PROP. LXX.

This is Prop. 62. in the Greek text; Prop. 78. in that text only a particular caſe of it, and is therefore omitted.

Dr Gregory, in the demonſtration of Prop. 62. cites the Prop. Dat. to prove that the ratio of the figure AEB to t parallelogram AH is given; whereas this was ſhewn a fe lines before; and beſides, the 49. Prop. is not applicable theſe two figures, becauſe AH is not given in ſpecies, but by the ſtep for which the citation is brought, proved to be giv in ſpecies.

PROP. LXXIII.

Prop. 83. in the Greek text is neither well enunciated n demonſtrated. The 73. which in this edition is put in place it, is really the ſame, as will appear by conſidering [See Gregory's Edition] that A, B, γ, E in the Greek text four proportionals; and that the Propoſition is to ſhew th Δ, which has a given ratio to E, is to γ, as B is to ſtraight line to which A has a given ratio; or, by inverſio that γ is to Δ, as a ſtraight line to which A has a giv ratio is to B; that is, if the proportionals be placed in th order, viz. γ, E, A, B, that the firſt γ is to Δ to which t ſecond E has a given ratio, as a ſtraight line to which t third A has a given ratio is to the fourth B; which is t enunciation of this 73. and was thus changed that it mig be made like to that of Prop. 72. in this edition, which

e 82. in the Greek text: And the demonſtration of Prop. 3. is the ſame with that of Prop. 72. only making uſe of rop. 23. inſtead of Prop. 22. of Book 5. of the Elements.

PROP. LXXVII.

This is put in place of Prop. 79. in the Greek text, which not a Datum, but a Theorem premiſed as a Lemma to rop. 80. in that text: And Prop. 79. is made Cor. 1. to rop. 77. in this Edition. Cl. Hardy, in his Edition of the ata, takes notice that, in Prop. 80. of the Greek text, the rallel KL in the figure of Prop. 77. in this edition muſt eet the circumference, but does not demonſtrate it, which done here at the end of Cor. 3. of Prop. 77. in the conruction for finding a triangle ſimilar to ABC.

PROP. LXXVIII.

The demonſtration of this, which is Prop. 80. in the Greek, rendered a good deal ſhorter by help of Prop. 77.

PROP. LXXIX. LXXX. LXXXI.

Theſe are added to Euclid's Data, as Propoſitions which e often uſeful in the ſolution of Problems.

PROP. LXXXII.

This, which is Prop. 60. in the Greek text, is placed before e 83. and 84. which in the Greek are the 58. and 59. beuſe the demonſtration of theſe two in this Edition is dedud from that of Prop. 82. from which they naturally follow.

PROP. LXXXVIII. XC.

Dr Gregory, in his preface to Euclid's works, which he bliſhed at Oxford in 1703, after having told that he had pplied the defects of the Greek text of the Data in innuerable places from ſeveral manuſcripts, and corrected Cl. ardy's tranſlation by Mr Bernard's, adds, that the 86. Theem "or Propoſition" ſeemed to be remarkably vitiated, t which could not be reſtored by help of the manuſcripts; en he gives three different tranſlations of it in Latin, acrding to which he thinks it may be read; the two firſt have diſtinct meaning, and the third, which he ſays is the beſt, o' it contains a true Propoſition, which is the 90. in this e-

dition,

dition, has no connexion in the leaſt with the Greek t
And it is ſtrange that Dr Gregory did not obſerve, tha
Prop. 86. was changed into this, the demonſtration of
86. muſt be cancelled, and another put in its place: But
truth is, both the enunciation and the demonſtration
Prop. 86. are quite entire and right, only Prop. 87. wh
is more ſimple, ought to have been placed before it; and
deficiency which the Doctor juſtly obſerves to be in this p
of Euclid's Data, and which no doubt is owing to the ca
leſſneſs and ignorance of the Greek editors, ſhould h
been ſupplied, not by changing Prop. 86. which is both
tire and neceſſary, but by adding the two Propoſitions wh
are the 88. and 90. in this edition.

PROP. XCVIII. C.

Theſe were communicated to me by two excellent Geo
ters, the firſt of them by the Right Honourable the E
Stanhope, and the other by Dr Matthew Stewart; to wh
I have added the demonſtrations.

Tho' the order of the Propoſitions has been in many p
ces changed from that in former editions, yet this will be
little diſadvantage, as the ancient Geometers never cite
Data, and the moderns very rarely.

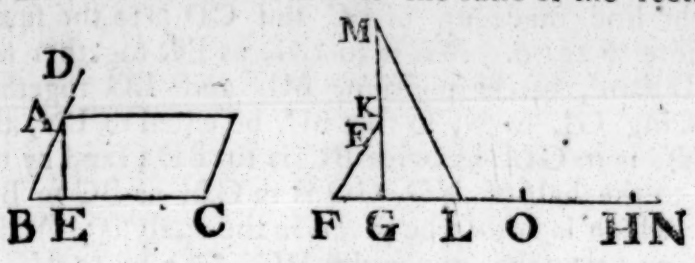
AS that part of the compoſition of a Problem which
its conſtruction may not be ſo readily deduced fr
the analyſis by beginners; for their ſake the following
ample is given in which the derivation of the ſeveral parts
the conſtruction from the analyſis is particularly ſhewn, t
they may be aſſiſted to do the like in other Problems.

PROBLEM.

Having given the magnitude of a parallelogram, the an
of which ABC is given, and alſo the exceſs of the ſquar
its ſide BC above the ſquare of the ſide AB; to find its ſi
and deſcribe it.

The analyſis of this is the ſame with the demonſtrat
of the 87. Prop. of the Data, and the conſtruction tha
given of the Problem at the end of that Propoſition is t
derived from the analyſis.

Let EFG be equal to the given angle ABC, and becauſe in the analyſis it is ſaid that the ratio of the rectangle AB, BC to the parallelogram AC is given by the 62. Prop. Dat. therefore, from a point in FE, the perpendicular EG is drawn to FG, as the ratio of FE to EG is the ratio of the rectangle



AB, BC to the parallelogram AC by what is ſhewn at the end of Prop. 62. Next, the magnitude of AC is exhibited by making the rectangle EG, GH equal to it; and the given exceſs of the ſquare BC above the ſquare of BA, to which exceſs the rectangle CB, BD is equal, is exhibited by the rectangle HG, GL: Then in the analyſis, the rectangle AB BC is ſaid to be given, and this is equal to the rectangle FE, GH, becauſe the rectangle AB, BC is to the parallelogram AC, as (FE to EG, that is, as the rectangle) FE, GH to EG, GH; and the parallelogram AC is equal to the rectangle EG, GH, therefore the rectangle AB, BC is equal to FE, GH: And conſequently the ratio of the rectangle CB, BD, that is, of the rectangle HG, GL, to AB, BC, that is, of the ſtraight line DB to BA, is the ſame with the ratio (of the rectangle GL, GH to FE, GH, that is) of the ſtraight line GL to FE, which ratio of DB to BA is the next thing ſaid to be given in the analyſis: From this it is plain that the ſquare of FE is to the ſquare of GL, as the ſquare of BA, which is equal to the rectangle BC, CD, is to the ſquare of BD: The ratio of which ſpaces is the next thing ſaid to be given: And from this it follows that four times the ſquare of FE is to the ſquare of GL, as four times the rectangle BC, CD is to the ſquare of BD; and, by compoſition, four times the ſquare of FE together with the ſquare of GL, is to the ſquare of GL, as four times the rectangle BC, CD together with the ſquare of BD, is to the ſquare of BD, that is (8. 6.) as the ſquare of the ſtraight lines BC, CD taken together is to the ſquare of BD, which ratio is the next thing ſaid to be given in the analyſis: And becauſe four times the ſquare of FE and the ſquare of GL are to be added together; therefore in the perpendicular

EG there is taken KG equal to FE, and MG equal to t
double of it, becauſe thereby the ſquares of MG, GL, th
is, joining ML, the ſquare of ML is equal to four times t
the ſquare of FE and to the ſquare of GL: And becauſe t
ſquare of ML is to the ſquare of GL, as the ſquare of t
ſtraight line made up of BC and CD is to the ſquare of B
therefore (22. 6.) ML is to LG, as BC together with CD
to BD; and, by compoſition, ML and LG together, that
producing GL to N, ſo that ML be equal to LN, the ſtraig
line NG is to GL, as twice BC is to BD; and by taking G
equal to the half of NG, GO is to GL, as BC to BD, the r
tio of which is ſaid to be given in the analyſis: And from th
it follows, that the rectangle HG, GO is to HG, GL,
the ſquare of BC is to the rectangle CB, BD which is equ
to the rectangle HG, GL; and therefore the ſquare of BC
equal to the rectangle HG, GO; and BC is conſequent
found by taking a mean proportional betwixt HG and G
as is ſaid in the conſtruction: And becauſe it was ſhewn th
GO is to GL, as BC to BD, and that now the three firſt a
found, the fourth BD is found by 12. 6. It was likewi
ſhewn that LG is to FE, or GK, as DB to BA, and t
three firſt are now found, and thereby the fourth BA
Make the angle ABC equal to EFG, and complete the para
lelogram of which the ſides are AB, BC, and the conſtru
tion is finiſhed; the reſt of the compoſition contains the d
monſtration.

AS the Propoſitions from the 13. to the 28. may be thoug
by beginners to be leſs uſeful than the reſt, becauſe th
cannot ſo readily ſee how they are to be made uſe of in t
ſolution of Problems; on this account the two following Pr
blems are added, to ſhew that they are equally uſeful with th
other Propoſitions, and from which it may be eaſily judg
that many other Problems depend upon theſe Propoſitions.

PROBLEM I.

TO find three ſtraight lines ſuch, that the ratio of th
firſt to the ſecond is given; and if a given ſtraig
line be taken from the ſecond, the ratio of the remainde
to the third is given; alſo the rectangle contained by th
firſt and third is given.

Le

Let AB be the first straight line, CD the second, and EF the third: And because the ratio of AB to CD is given, and that if a given straight line be taken from CD, the ratio of the remainder to EF is given; therefore [a] the excess of the first AB above a given straight line has a given ratio to the third EF: Let BH be that given straight line; therefore AH, the excess of AB above it, has a given ratio to EF; and consequently [b] the rectangle BA, AH has a given ratio to the rectangle EB, EF, which last rectangle is given by the hypothesis; therefore [c] the rectangle BA, AH is given, and BH the excess of its sides is given; wherefore the sides AB, AH are given [d]: And because the ratios of AB to CD, and of AH to EF are given, CD and EF are [e] given.

A H B

C G D

E F

K NM L O

a. 24. Dat.

b. 1. 9.

c. 2. Dat.

d 85. Dat.

The Composition.

Let the given ratio of KL to KM be that which AB is required to have to CD; and let DG be the given straight line which is to be taken from CD, and let the given ratio of KM to KN be that which the remainder must have to EF; also, let the given rectangle NK, KO be that to which the rectangle AB, EF is required to be equal: Find the given straight line BH which is to be taken from AB, which is done, as plainly appears from Prop. 24. Dat. by making as KM to KL, so GD to HB. To the given straight line BH apply [e] a rectangle equal to LK, KO exceeding by a square, and let BA, AH be its sides: Then is AB the first of the straight lines required to be found, and by making as LK to KM, so AB to DC, DC will be the second: And lastly, make as KM to KN, so CG to EF, and EF is the third.

e. 29. 6.

For as AB to CD, so is HB to GD, each of these ratios being the same with the ratio of LK to KM; therefore [f] AH to CG, as (AB to CD, that is, as) LK to KM; and as CG to EF, so is KM to KN; wherefore, ex aequali, as AH to EF, so is LK to KN: And as the rectangle BA, AH to the rectangle BA, EF, so is [g] the rectangle LK, KO to the rectangle KN, KO: And, by the construction, the rectangle BA, AH is equal to LK, KO: Therefore [h] the rectangle AB, EF is equal to the given rectangle NK, KO: And AB has to CD the given ratio of KL to KM; and from CD the given straight line GD being taken, the remainder CG has to EF the given ratio of KM to KN. Q. E. D.

f. 19. 5.

g. 1. 6.

h. 14. 5.

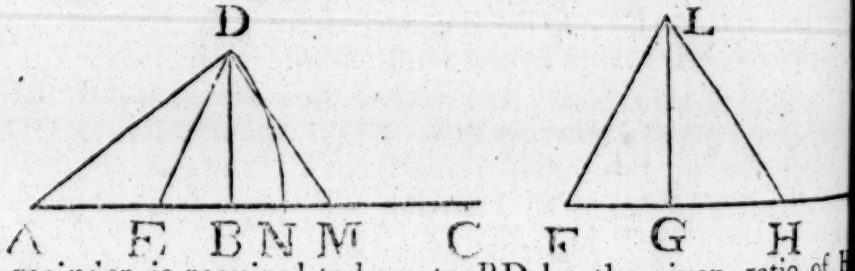
PROB. II.

TO find three ſtraight lines ſuch, that the ratio of t
firſt to the ſecond is given ; and if a given ſtraig
line be taken from the ſecond, the ratio of the remaind
to the third is given ; alſo the ſum of the ſquares of t
firſt and third is given.

Let AB be the firſt ſtraight line, BC the ſecond, and B
the third : And becauſe the ratio of AB to BC is given, a
that if a given ſtraight line be taken from BC, the ratio
a.24.Dat. the remainder to BD is given ; therefore [a] the exceſs of t
firſt AB above a given ſtraight line, has a given ratio to t
third BD : Let AE be that given ſtraight line, therefore t
remainder EB has a given ratio to BD : Let BD be plac
b.44.Dat. at right angles to EB, and join DE ; then the triangle EBD i
given in ſpecies ; wherefore the angle BED is given : Let A
which is given in magnitude, be given alſo in poſition, as a
c.32.Dat. the point E, and the ſtraight line ED will be given [c] in po
tion : Join AD, and becauſe the ſum of the ſquares of A
d.47. 1. BD, that is [d], the ſquare of AD is given, therefore t
e.34.Dat. ſtraight line AD is given in magnitude ; and it is alſo give
in poſition, becauſe from the given point A it is drawn to t
ſtraight line ED given in poſition : Therefore the point D,
which the two ſtraight lines AD, ED given in poſition
f.28.Dat. one another, is given [f] : And the ſtraight line DB which is
g 33 Dat. right angles to AB is given [g] in poſition, and AB is given
poſition, therefore [f] the point B is given : And the points
h.29.Dat. D are given, wherefore [h] the ſtraight lines AB, BD are give
. 2. Dat. And the ratio of AB to BC is given, and therefore [i] BC
given.

The Compoſition.

Let the given ratio of FG to GH be that which AB is req
red to have to BC, and let HK be the given ſtraight l
which is to be taken from BC, and let the ratio which the



mainder is required to have to BD be the given ratio of H
to GL, and place GL at right angles to FH, and join LF, L

Next, as HG is to GF, ſo make HK to AE; produce AE to N, ſo that AN be the ſtraight line to the ſquare of which the ſum of the ſquares of AB, BD is required to be equal; and make the angle NED equal to the angle GFL; from the center A at the diſtance AN deſcribe a circle, and let its circumference meet ED in D, and draw DB perpendicular to AN, and DM making the angle BDM equal the angle GLH. Laſtly, produce BM to C, ſo that MC be equal to HK; then is AB the firſt, BC the ſecond, and BD the third of the ſtraight lines that were to be found.

For the triangles EBD, FGL, as alſo DBM, LGH being equiangular, as EB to BD, ſo is FG to GL; and as DB to BM, ſo is LG to GH; therefore, ex æquali, as EB to BM, ſo is (FG to GH, and ſo is) AE to HK or MC; wherefore[k], AB is to BC, as AE to HK, that is, as FG to GH, that is, in the given ratio; and from the ſtraight line BC taking MC, which is equal to the given ſtraight line HK, the remainder BM has to BD the given ratio of HG to GL; and the ſum of the ſquares of AB, BD is equal[d] to the ſquare of AD or AN, which is the given ſpace. Q. E. D.

k 12 5.

d 47. 1.

I believe it would be in vain to try to deduce the preceeding conſtruction from an algebraical ſolution of the Problem.

FINIS.

16 OC 65

THE

ELEMENTS

OF

PLAIN AND SPHERICAL

TRIGONOMETRY.

EDINBURGH:

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MD,CC,LXXII.

BRITISH MUSEUM

LEMMA I. Fig. 1.

ET ABC be a rectilineal angle, if about the point B as a center, and with any distance BA, a circle be described, meeting BA, BC, the straight lines including the angle ABC in A, C; the angle ABC will be to four right angles, as the arch AC to the whole circumference.

Produce AB till it meet the circle again in F, and thro' B draw DE perpendicular to AB, meeting the circle in D, E.

By 33. 6. Elem. the angle ABC is to a right angle ABD, as the arch AC to the arch AD; and quadrupling the consequents, the angle ABC will be to four right angles, as the arch AC to four times the arch AD, or to the whole circumference.

LEMMA II. Fig. 2.

ET ABC be a plain rectilineal angle as before: About B as a center with any two distances BD, BA, let two circles be described meeting BA, BC in D, E, A, C; the arch AC will be to the whole circumference of which it is an arch, as the arch DE is to the whole circumference of which it is an arch.

By Lemma 1. the arch AC is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; and by the same Lemma 1. the arch DE is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; therefore the arch AC is to the whole circumference of which it is an arch, as the arch DE to the whole circumference of which it is an arch.

DEFINITIONS. Fig. 3.

I.

ET ABC be a plain rectilineal angle; if about B as a center, with BA any distance, a circle ACF be described meeting BA, BC, in A, C; the arch AC is called the measure of the angle ABC.

II.

The circumference of a circle is supposed to be divided into 360

360 equal parts called degrees, and each degree into 60 equal parts called minutes, and each minute into 60 equal parts called ſeconds, &c. And as many degrees, minutes, ſeconds, &c. as are contained in any arch, of ſo many degrees, minutes, ſeconds, &c. is the angle, of which that arch is the meaſure, ſaid to be

Cor. Whatever be the radius of the circle of which the meaſure of a given angle is an arch, that arch will contain the ſame number of degrees, minutes, ſeconds, &c. as is manifeſt from Lemma 2.

III.

Let AB be produced till it meet the circle again in F, the angle CBF, which, together with ABC, is equal to two right angles, is called the *Supplement* of the angle ABC.

IV.

A ſtraight line CD drawn thro' C, one of the extremities of the arch AC, perpendicular upon the diameter paſſing thro the other extremity A, is called the *Sine* of the arch AC, of the angle ABC, of which it is the meaſure.

Cor. The *Sine* of a quadrant, or of a right angle, is equal to the radius.

V.

The ſegment DA of the diameter paſſing thro' A, one extremity of the arch AC between the ſine CD, and that extremity is called the *verſed Sine* of the arch AC, or angle ABC.

VI.

A ſtraight line AE touching the circle at A, one extremity of the arch AC, and meeting the diameter BC paſſing through the other extremity C in E, is called the *Tangent* of the arch AC, or of the angle ABC.

VII.

The ſtraight line BE between the center and the extremity of the tangent AE, is called the *Secant* of the arch AC, or angle ABC.

Cor. to Def. 4. 6. 7. The Sine, Tangent, and Secant of an angle ABC, are likewiſe the Sine, Tangent, and Secant of its Supplement CBF.

It is manifeſt from Def. 4. that CD is the Sine of the angle CBF. Let CB be produced till it meet the circle again in G, and it is manifeſt that AE is the *Tangent*, and BE the *Secant*, of the angle ABG or EBF, from Def. 6. 7.

Co

Cor. to Def. 4. 5. 6. 7. The *Sine*, *verſed Sine*, *Tangent*, and *Secant*, of any arch which is the meaſure of any given angle ABC, is to the Sine, verſed Sine, Tangent, and Secant, of any other arch which is the meaſure of the ſame angle, as the Radius of the firſt is to the Radius of the ſecond. Fig. 4.

Let AC, MN be meaſures of the angles ABC, according to Def. 1. CD the Sine, DA the verſed Sine, AE the Tangent, and BE the Secant of the arch AC, according to Def. 4. 5. 6. 7. and NO the Sine, OM the verſed Sine, MP the Tangent, and BP the Secant of the arch MN, according to the ſame definitions. Since CD, NO, AE, MP are parallel, CD is to NO as the radius CB to the radius NB, and AE to MP as AB to BM, and BC or BA to BD as BN or BM to BO; and, by converſion, DA to MO as AB to MB. Hence the Corollary is manifeſt; therefore, if the Radius be ſuppoſed to be divided into any given number of equal parts, the Sine, verſed Sine, Tangent, and Secant of any given angle, will each contain a given number of theſe parts; and, by Trigonometrical Tables, the length of the Sine, verſed Sine, Tangent, and Secant of any angle may be found in parts of which the radius contains a given number: and, vice verſa, a number expreſſing the length of the Sine, verſed Sine, Tangent, and Secant being given, the angle of which it is the Sine, verſed Sine, Tangent, and Secant may be found.

VIII. Fig. 3.

The difference of an angle from a right angle is called the complement of that angle. Thus, if BH be drawn perpendicular to AB, the angle CBH will be the complement of the angle ABC, or of CBF.

IX.

If HK be the Tangent, CL or DB, which is equal to it, the Sine, and BK the Secant of CBH, the complement of ABC, according to Def. 4. 6. 7. HK is called the Co-tangent, BD the Co-ſine, and BK the Co-ſecant of the angle ABC.

Cor. 1. The Radius is a mean proportional between the Tangent and Co-tangent.

For, ſince HK, BA are parallel, the angles HKB, ABC will be equal, and the angles KHB, BAE are right; therefore the

the triangles BAE, KHB are ſimilar, and therefore AE to AB, as BH or BA to HK.

Cor. 2. The Radius is a mean proportional between the C ſine and Secant of any angle ABC.

Since CD, AE are parallel, BD is to BC or BA, as BA BE.

PROP. I. Fig. 5.

IN a right angled plain triangle, if the hpyothen be made radius, the ſides become the ſines of angles oppoſite to them; and if either ſide be m radius, the remaining ſide is the tangent of the angle poſite to it, and the hypothenuſe the ſecant of the ſa angle.

Let ABC be a right angled triangle; if the hypothen BC be made radius, either of the ſides AC will be the ſin the angle ABC oppoſite to it; and if either ſide BA be m radius, the other ſide AC will be the tangent of the a ABC oppoſite to it, and the hypothenuſe BC the ſecant of ſame angle.

About B as a center, with BC, BA for diſtances, let circles CD, EA be deſcribed, meeting BA, BC in D, E: S CAB is a right angle, BC being radius, AC is the ſine o angle ABC by Def. 4. and BA being radius, AC is the gent, and BC the ſecant of the angle ABC, by Def. 6. 7.

Cor. 1. Of the hypothenuſe a ſide and an angle of a angled triangle, any two being given, the third is alſo g

Cor. 2. Of the two ſides and an angle of a right angled angle, any two being given, the third is alſo given.

PROP. II. Fig. 6. 7.

THE ſides of a plain triangle are to one anothe the ſines of the angles oppoſite to them.

In right angled triangles this Prop. is manifeſt from 1. for, if the hypothenuſe be made radius, the ſides ar ſines of the angles oppoſite to them, and the radius ſine of a right angle (Cor. to Def. 4.) which is oppoſ the hypothenuſe.

In any oblique angled triangle ABC, any two ſides AB, AC will be to one another as the ſines of the angles ACB, ABC which are oppoſite to them.

From C, B draw CE, BD perpendicular upon the oppoſite ſides AB, AC produced, if need be. Since CEB, CDB are right angles, BC being radius, CE is the ſine of the angle CBA, and BD the ſine of the angle ACB; but the two triangles CAE, DAB have each a right angle at D and E; and likewiſe the common angle CAB; therefore they are ſimilar, and conſequently, CA is to AB, as CE to DB; that is, the ſides are as the ſines of the angles oppoſite to them.

Cor. Hence of two ſides, and two angles oppoſite to them, in a plain triangle, any three being given, the fourth is alſo given.

PROP. III. Fig. 8.

IN a plain triangle, the ſum of any two ſides is to their difference, as the tangent of half the ſum of the angles at the baſe, to the tangent of half their difference.

Let ABC be a plain triangle, the ſum of any two ſides AB, AC will be to their difference as the tangent of half the ſum of the angles at the baſe ABC, ACB to the tangent of half their difference.

About A as a center, with AB the greater ſide for a diſtance, let a circle be deſcribed, meeting AC produced in E, F, and BC in D; join DA, EB, FB; and draw FG parallel to BC, meeting EB in G.

The angle EAB (32. 1.) is equal to the ſum of the angles at the baſe, and the angle EFB at the circumference is equal to the half of EAB at the center (20. 3.); therefore EFB is half the ſum of the angles at the baſe; but the angle ACB (32. 1.) is equal to the angles CAD and ADC, or ABC together; therefore FAD is the difference of the angles at the baſe, and FBD at the circumference, or BFG, on account of the parallels FG, BD, is the half of that difference; but ſince the angle EBF in a ſemicircle is a right angle (1. of this) FB being radius, BE, BG, are the tangents of the angles EFB, BFG; but it is manifeſt that EC is the ſum of the ſides BA, AC, and CF their difference; and ſince BC, FG are parallel (2. 6.) EC is to CF, as EB to BG; that is, the ſum of the

ſides

ſides is to their difference, as the tangent of half the ſum o the angles at the baſe to the tangent of half their difference.

PROP. IV. Fig. 18.

IN any plain triangle BAC, whoſe two ſides are BA AC and baſe BC, the leſs of the two ſides, which le be BA, is to the greater AC as the radius is to the tan gent of an angle; and the radius is to the tangent o the exceſs of this angle above half a right angle as th tangent of half the ſum of the angles B and C at th baſe, is to the tangent of half their difference.

At the point A, draw the ſtraight line EAD perpendicula to BA; make AE, AF, each equal to AB, and AD to AC join BE, BF, BD, and from D, draw DG perpendicular upo BF. And becauſe BA is at right angles to EF, and EA, AB AF are equal, each of the angles EBA, ABF is half a righ angle, and the whole EBF is a right angle; alſo (4. 1. El.) E is equal to BF. And ſince EBF, FGD are right angles, EB i parallel to GD, and the triangles EBF, FGD are ſimilar therefore EB is to BF as DG to GF, and EB being equal t BF, FG muſt be equal to GD. And becauſe BAD is a righ angle, BA the leſs ſide is to AD or AC the greater, as the ra dius is to the tangent of the angle ABD; and becauſe BGD i a right angle, BG is to GD or GF as the radius is to the tan gent of GBD, which is the exceſs of the angle ABD abov ABF half a right angle. But becauſe EB is parallel to GD, BG is to GF as ED is to DF, that is, ſince ED is the ſum of th ſides BA, AC and FD their difference, (3. of this,) as the tan gent of half the ſum of the angles B, C, at the baſe to the tangen of half their difference. Therefore, in any plain triangle, &c Q. E. D.

PROP. V. Fig. 9. and 10.

IN any triangle, twice the rectangle contained by an two ſides is to the difference of the ſum of the ſquare of theſe two ſides, and the ſquare of the baſe, as the ra dius is to the co-ſine of the angle included by the tw ſides.

Let ABC be a plain triangle, twice the rectangle ABC con tained by any two ſides BA, BC is to the difference of the ſum

of the ſquares of BA, BC, and the ſquare of the baſe AC, as the radius to the Co-ſine of the angle ABC.

From A, draw AD perpendicular upon the oppoſite ſide BC; then (by 12. and 13. 2. El.) the difference of the ſum of the ſquares of AB, BC, and the ſquare of the baſe AC, is equal to twice the rectangle CBD; but twice the rectangle CBA is to twice the rectangle CBD, that is, to the difference of the ſum of the ſquares of AB, BC, and the ſquare of AC, (1. 6.) as AB to BD; that is, by Prop. 1. as radius to the Sine of BAD, which is the complement of the angle ABC, that is, as radius to the co-ſine of ABC.

PROP. VI. FIG. 11.

IN any triangle ABC, whoſe two ſides are AB, AC, and baſe BC; the rectangle contained by half the perimeter, and the exceſs of it above the baſe BC, is to the rectangle contained by the ſtraight lines, by which the half of the perimeter exceeds the other two ſides AB, AC, as the ſquare of the radius is to the ſquare of the tangent of half the angle BAC oppoſite to the baſe.

Let the angles BAC, ABC be biſected by the ſtraight lines AG, BG; and, producing the ſide AB, let the exterior angle CBH be biſected by the ſtraight line BK, meeting AG in K; and from the points G, K, let there be drawn perpendicular upon the ſides the ſtraight lines GD, GE, GF, KH, KL, KM. Since therefore (4. 4.) G is the center of the circle inſcribed in the triangle ABC, GD, GF, GE will be equal, and AD will be equal to AE, BD to BF, and CE to CF. In like manner KH, KL, KM will be equal, and BH will be equal to BM, and AH to AL, becauſe the angles HBM, HAL are biſected by the ſtraight lines BK, KA: And becauſe in the triangles KCL, KCM, the ſides LK, KM are equal, KC is common and KLC, KMC are right angles, CL will be equal to CM: Since therefore BM is equal to BH, and CM to CL; BC will be equal to BH and CL together; and, adding AB and AC together, AB, AC, and BC will together be equal to AH and AL together: But AH, AL are equal: Wherefore each of them is equal to half the perimeter of the triangle ABC: But ſince AD, AE are equal, and BD, BF, and alſo CE, CF, AB together with FC, will be equal to half the perimeter of the triangle to which AH or AL was ſhewn to be equal; taking away therefore the common AB, the remainder

der FC will be equal to the remainder BH: In the ſame ma
ner is it demonſtrated, that BF is equal to CL: And ſince t
points B, D, G, F, are in a circle, the angle DGF will be
qual to the exterior and oppoſite angle FBH, (22. 3.); whe
fore their halves BGD, HBK will be equal to one anoth
The right angled triangles BGD, HBK will therefore be
quiangular, and GD will be to BD, as BH to HK, and t
rectangle contained by GD, HK will be equal to the rect
gle DBH or BFC: But ſince AH is to HK, as AD to D
the rectangle HAD (22. 6.) will be to the rectangle contain
by HK, DG, or the rectangle BFC, (as the ſquare of AD
to the ſquare of DG, that is) as the ſquare of the radius
the ſquare of the tangent of the angle DAG, that is, the h
of BAC: But HA is half the perimeter of the triangle AB
and AD is the exceſs of the ſame above HD, that is, above t
baſe BC; but BF or CL is the exceſs of HA or AL abo
the ſide AC, and FC, or HB is the exceſs of the ſame HA
bove the ſide AB; therefore the rectangle contained by h
the perimeter, and the exceſs of the ſame above the ba
viz. the rectangle HAD, is to the rectangle contained by t
ſtraight lines by which the half of the perimeter exceeds t
other two ſides, that is, the rectangle BFC, as the ſquare of t
radius is to the ſquare of the tangent of half the angle BA
oppoſite to the baſe. Q. E. D.

PROP. VII. FIG. 12. 13.

IN a plain triangle, the baſe is to the ſum of the ſide
as the difference of the ſides is to the ſum or dif
rence of the ſegments of the baſe made by the perpen
cular upon it from the vertex, according as the ſquare
the greater ſide is greater or leſs than the ſum of t
ſquares of the leſſer ſide and the baſe.

Let ABC be a plane triangle; if from A the vertex
drawn a ſtraight line AD perpendicular upon the baſe B
the baſe BC will be to the ſum of the ſides BA, AC, as t
difference of the ſame ſides is to the ſum or difference of t
ſegments CD, BD, according as the ſquare of AC the gre
er ſide is greater or leſs than the ſum of the ſquares of t
leſſer ſide AB, and the baſe BC.

About A as a center, with AC the greater ſide for a
ſtance, let a circle be deſcribed meeting AB produced in
F, and CB in G: It is manifeſt that FB is the ſum, and

the difference of the ſides; and ſince AD is perpendicular to BC, GD, CD will be equal; conſequently GB will be equal to the ſum or difference of the ſegments CD, BD, according as the perpendicular AD meets the baſe, or the baſe produced; that is, (by Conv. 12. and 13. 2.) according as the ſquare of AC is greater or leſs than the ſum of the ſquares of AB, BC: But (by 35. 3.) the rectangle CBG is equal to the rectangle EBF; that is, (16. 6.) BC is to BF, as BE is to BG: that is, the baſe is to the ſum of the ſides, as the difference of the ſides is to the ſum or difference of the ſegments of the baſe made by the perpendicular from the vertex, according as the ſquare of the greater ſide is greater or leſs than the ſum of the ſquares of the leſſer ſide and the baſe. Q. E. D.

PROP. VIII. PROB. Fig. 14.

THE ſum and difference of two magnitudes being given, to find them.

Half the given ſum added to half the given difference, will be the greater, and half the difference ſubtracted from half the ſum, will be the leſs.

For, let AB be the given ſum, AC the greater, and BC the leſs. Let AD be half the given ſum; and to AD, DB, which are equal, let DC be added, then AC will be equal to BD, and DC together; that is, to BC, and twice DC; conſequently twice DC is the difference, and DC half that difference; but AC the greater is equal to AD, DC; that is, to half the ſum added to half the difference, and BC the leſs is equal to the exceſs of BD, half the ſum above DC half the difference. Q. E. F.

SCHOLIUM.

Of the ſix parts of a plain triangle (the three ſides and three angles) any three being given, to find the other three is the buſineſs of plain Trigonometry; and the ſeveral caſes of that Problem may be reſolved by means of the preceeding Propoſitions, as in the two following, with the Tables annexed. In theſe, the ſolution is expreſſed by a fourth proportional to three given lines; but if the given parts be expreſſed by numbers from Trigonometrical Tables, it may be obtained arithmetically by the common Rule of Three.

Note. In the tables the following abbreviations are uſed. R, is put for the Radius; T, for Tangent; and S, for Sine. Degrees, minutes, ſeconds, &c. are written in this manner; 30° 25′ 13″ &c. which ſignifies 30 degrees, 25 minutes, 13 ſeconds, &c.

SOLUTION of the CASES of right-angled TRIANGLES.

GENERAL PROPOSITION.

IN a right angled triangle, of the three ſides and three angles, any two being given beſides the right angle, the other three may be found, except when the two acute angles are given, in which caſe the ratios of the ſides are only given, being the ſame with the ratios of the ſines of the angles oppoſite to them.

It is manifeſt from 47. 1. that of the two ſides and hypothenuſe any two be given the third may alſo be found. It is alſo manifeſt from 32. 1. that if one of the acute angles of a right angled triangle be given, the other is alſo given, for it is the complement of the former to a right angle.

If two angles of any triangle be given, the third is alſo given being the ſupplement of the two given angles to two right angles.

Fig. 15. The other caſes may be reſolved by help of the preceeding propoſitions, as in the following table.

	GIVEN.	SOUGHT.	
1	Two ſides, AB AC.	The angles B, C.	AB : AC : : R : T, B. of which C is the complement
2	AB, BC, a ſide and the hypothenuſe.	The angles B, C.	BC : BA : : R : S, C, of which B is the complement
3	AB, B, a ſide and an angle.	The other ſide AC.	R : T, B : : BA : AC.
4	AB and B, a ſide and an angle.	The hypothenuſe BC.	S, C : R : : BA : BC.
5	BC and B, the hypothenuſe and an angle.	The ſide AC.	R : S, B : : BC : CA.

Theſe five caſes are reſolved by Prop. 1.

SOLUTION of the CASES of oblique-angled TRIANGLES.

GENERAL PROPOSITION.

IN an oblique-angled triangle, of the three ſides and three angles, any three being given, the other three may be found, except when the three angles are given; in which caſe, the ratios of the ſides are only given, being the ſame with the ratios of the ſines of the angles oppoſite to them.

	GIVEN.	SOUGHT.	
1	A, B, and therefore C, and the ſide AB.	BC, AC.	S, C : S, A : : AB : BC, and alſo S, C : S, B : : AB : AC. (2.) Fig16.17.
2	AB, AC, and B, two ſides and an angle oppoſite to one of them.	The angles A and C.	AC : AB : : S, B : S, C. (2.) This caſe admits of two ſolutions; for C may be greater or leſs than a quadrant. (Cor. to Def. 4.)
3	AB, AC, and A, two ſides, and the included angle.	The angles B and C.	AB + AC : AB—AC : : T, C + B. : T, C—B : (3.) and the ſum and difference of the angles C, B, being given, each of them is given. (7.) Otherwiſe. FIG. 18. BA : AC : : R : T, ABC, and alſo R : T, ABC—45° : : T, B+C : T, B—C : (4.) therefore B and C are given as before. (7.)

	Given.	Sought.	
4	AB, BC, CA, the three sides.	A, B, C, the three angles.	2 AC × CB : $AC^q + CB^q$ —AB^q : : R : CoS, C. If AC^q + CB^q be greater than AB^q Fig. 16. 2 AC × CB : AB^q—AC^q—CB^q : : R : CoS, C. If AB^q be greater than $AC^q + CB^q$ Fig. 17. (4.) Otherwiſe. Let AB + BC + AC = 2 P $P \times P-AB : P-AC \times P-BC$: : R^q : T^q, $\frac{1}{2}$ C, and hence C is known. (5.) Otherwiſe. Let AD be perpendicular to BC. 1. If AB^q be leſs than $AC^q + CB^q$. Fig. 16. BC : BA + AC : : BA—AC : BD —DC, and BC the ſum of BD, DC is given; therefore each of them is given. (7.) 2 If AB^q be greater than $AC^q + CB^q$. Fig. 17. BC : BA + AC : : BA—AC : BD + DC; and BC the difference of BD, DC is given, therefore each of them is given. (7.) And CA : CD : : R : CoS, C. (1.) and C being found, A and B are found by Caſe 2. or 3.

SPHERICAL

SPHERICAL TRIGONOMETRY.

DEFINITIONS.

I.

THE pole of a circle of the ſphere is a point in the ſuperficies of the ſphere, from which all ſtraight lines drawn to the circumference of the circle are equal.

II.

A great circle of the ſphere is any whoſe plane paſſes through the center of the ſphere, and whoſe center therefore is the ſame with that of the ſphere.

III.

A ſpherical triangle is a figure upon the ſuperficies of a ſphere comprehended by three arches of three great circles, each of which is leſs than a ſemicircle.

IV.

A ſpherical angle is that which on the ſuperficies of a ſphere is contained by two arches of great circles, and is the ſame with the inclination of the planes of theſe great circles.

PROP. I.

GREAT circles biſect one another.

As they have a common center their common ſection will be a diameter of each which will biſect them.

PROP. II. FIG. 1.

THE arch of a great circle betwixt the pole and circumference of another is a quadrant.

Let ABC be a great circle and D its pole; if a great circle DC paſs through D, and meet ABC in C, the arch DC will be a quadrant.

Let the great circle CD meet ABC again in A, and let AC be the common ſection of the great circles, which will

paſs thro' E the center of the ſphere: Join DE, DA, DC. By Def. 1. DA, DC are equal, and AE, EC are alſo equal, and DE is common; therefore (8. 1.) the angles DEA, DEC are equal; wherefore the arches DA, DC are equal, and conſequently each of them is a quadrant. Q. E. D.

PROP. III. FIG. 2.

IF a great circle be deſcribed meeting two great circles AB, AC paſſing thro' its pole A in B, C, the angle at the center of the ſphere upon the circumference BC, is the ſame with the ſpherical angle BAC, and the arch BC is called the meaſure of the ſpherical angle BAC.

Let the planes of the great circles AB, AC interſect one another in the ſtraight line AD paſſing thro' D their common center; join DB, DC.

Since A is the pole of BC, AB, AC will be quadrants, and the angles ADB, ADC right angles; therefore (6. Def. 11.) the angle CDB is the inclination of the planes of the circles AB, AC; that is, (Def. 4.) the ſpherical angle BAC. Q. E. D.

COR. If thro' the point A, two quadrants AB, AC, be drawn, the point A will be the pole of the great circle BC, paſſing thro' their extremities B, C.

Join AC, and draw AE a ſtraight line to any other point E in BC; join DE: Since AC, AB are quadrants, the angles ADB, ADC are right angles, and AD will be perpendicular to the Plane of BC: Therefore the angle ADE is a right angle, and AD, DC are equal to AD, DE each to each; therefore AE, AC are equal, and A is the pole of BC, by Def. 1. Q. E. D.

PROP. IV. FIG. 3.

IN Iſoſceles ſpherical triangles, the angles at the baſe are equal.

Let ABC be an Iſoſceles triangle, and AC, CB the equal ſides; the angles BAC, ABC, at the baſe AB, are equal.

Let

Let D be the center of the ſphere, and join DA, DB, C; in DA take any point E, from which draw, in the plane DC, the ſtraight line EF at right angles to ED meeting CD F, and draw, in the plane ADB, EG at right angles to the me ED; therefore the rectilineal angle FEG is (6. Def. 11.) e inclination of the planes ADC, ADB, and therefore is e ſame with the ſpherical angle BAC: From F draw FH rpendicular to DB, and from H draw, in the plane ADB, e ſtraight line HG at right angles to HD meeting EG in and join GF. Becauſe DE is at right angles to EF and G, it is perpendicular to the plane FEG, (4. 11.) and there- e the plane FEG is perpendicular to the plane ADB, in hich DE is: (18. 11.) In the ſame manner the plane FHG perpendicular to the plane ADB; and therefore GF the mmon ſection of the planes FEG, FHG is perpendicular the plane ADB; (19. 11.) and becauſe the angle FHG is e inclination of the planes BDC, BDA, it is the ſame with e ſpherical angle ABC; and the ſides AC, CB of the ſphe- al triangle being equal, the angles EDF, HDF, which nd upon them at the center of the ſphere, are equal; and the triangles EDF, HDF the ſide DF is common, and the gles DEF, DHF are right angles; therefore EF, FH are e- al; and in the triangles FEG, FHG the ſide GF is com- n and the ſides EG, GH will be equal by the 47. 1. and erefore the angle FEG is equal to FHG; (8. 1.) that is, e ſpherical angle BAC is equal to the ſpherical angle ABC.

PROP. V. FIG. 3.

F, in a ſpherical triangle ABC, two of the angles BAC, ABC be equal, the ſides BC, AC oppoſite to them, e equal.

Read the conſtruction and demonſtration of the preceed- propoſition, unto the words, "and the ſides AC, CB, &c." and the reſt of the demonſtration will be as follows,

And the ſpherical angles BAC ABC being equal, the tilineal angles FEG, FHG, which are the ſame with them, equal; and in the triangles FGE, FGH the angles at G right angles, and the ſide FG oppoſite to two of the equal

angles is common; therefore (26. 1.) EF is equal to FH; an in the right-angled triangles DEF, DHF the ſide DF is com mon; wherefore (47. 1.) ED is equal to DH, and the angle EDF, HDF are therefore equal, (4. 1.) and conſequently th ſides AC, BC of the ſpherical triangle are equal.

PROP. VI. FIG. 4.

ANY two ſides of a ſpherical triangle are greater tha the third.

Let ABC be a ſpherical triangle, any two ſides AB, B will be greater than the other ſide AC.

Let D be the center of the ſphere; join DA, DB, DC.

The ſolid angle at D is contained by three plane angl ADB, ADC, BDC; and by 20. 11. any two of them AD BDC are greater than the third ADC; that is, any two ſid AB, BC of the ſpherical triangle ABC, are greater than t third AC.

PROP. VII. FIG. 4.

THE three ſides of a ſpherical triangle are leſs than circle.

Let ABC be a ſpherical triangle as before, the three ſi AB, BC, AC are leſs than a circle.

Let D be the center of the ſphere: The ſolid angle at D contained by three plane angles BDA, BDC, ADC, which gether are leſs than four right angles, (21. 11.) theref the ſides AB, BC, AC together, will be leſs than four q drants; that is, leſs than a circle.

PROP. VIII. FIG. 5.

IN a ſpherical triangle the greater angle is oppoſite the greater ſide; and converſely.

Let ABC be a ſpherical triangle, the greater angle A is poſed to the greater ſide BC.

Let the angle BAD be made equal to the angle B, then BD, DA will be equal, (5. of this) and therefore A

DC are equal to BC; but AD, DC are greater than AC, (6. of this.) therefore BC is greater than AC, that is, the greater angle A is oppoſite to the greater ſide BC. The converſe is demonſtrated as Prop. 19. 1. El. Q. E. D.

PROP. IX. FIG. 6.

IN any ſpherical triangle ABC, if the ſum of the ſides AB, BC be greater, equal, or leſs than a ſemicircle, the internal angle at the baſe AC will be greater, equal, or leſs than the external and oppoſite BCD; and therefore the ſum of the angles A and ACB will be greater, equal, or leſs than two right angles.

Let AC, AB produced meet in D.

1. If AB, BC be equal to a ſemicircle, that is, to AD, BC, BD will be equal, that is, (4. of this) the angle D, or the angle A will be equal to the angle BCD.

2. If AB, BC together be greater than a ſemicircle, that is, greater than ABD, BC will be greater than BD; and therefore (8. of this) the angle D, that is, the angle A, is greater than the angle BCD.

3. In the ſame manner is it ſhewn, that if AB, BC together be leſs than a ſemicircle, the angle A is leſs than the angle BCD. And ſince the angles BCD, BCA are equal to two right angles, if the angle A be greater than BCD, A and ACB together will be greater than two right angles. If A be equal to BCD, A and ACB together will be equal to two right angles; and if A be leſs than BCD, A and ACB will be leſs than two right angles. Q. E D.

PROP. X. FIG. 7.

IF the angular points A, B, C of the ſpherical triangle ABC be the poles of three great circles, theſe great circles by their interſections will form another triangle FDE, which is called ſupplemental to the former; that is, the ſides FD, DE, EF are the ſupplements of the meaſures

meaſures of the oppoſite angles C, B, A, of the triangle ABC, and the meaſures of the angles F, D, E of the triangle FDE, will be the ſupplements of the ſides AC, BC, BA, in the triangle ABC.

Let AB produced meet DE, EF in G, M, and AC meet FD, FE in K, L, and BC meet FD, DE in N, H.

Since A is the pole of FE, and the circle AC paſſes through A, EF will paſs through the pole of AC, (13. 15. 1. Th.) and ſince AC paſſes thro' C, the pole of FD, FD will paſs through the pole of AC; therefore the pole of AC is in the point F, in which the arches DF, EF interſect each other. In the ſame manner, D is the pole of BC, and E the pole of AB.

And ſince F, E are the poles of AL, AM, FL and EM are quadrants, and FL, EM together, that is, FE and ML together, are equal to a ſemicircle. But ſince A is the pole of ML, ML is the meaſure of the angle BAC, conſequently, FE is the ſupplement of the meaſure of the angle BAC. In the ſame manner, ED, DF are the ſupplements of the meaſures of the angles ABC, BCA.

Since likewiſe CN, BH, are quadrants, CN, BH together, that is, NH, BC together, are equal to a ſemicircle; and ſince D is the pole of NH, NH is the meaſure of the angle FDE, therefore the meaſure of the angle FDE is the ſupplement of the ſide BC. In the ſame manner, it is ſhewn that the meaſures of the angles DEF, EFD are the ſupplements of the ſides AB, AC, in the triangle ABC. Q. E. D.

PROP. XI. FIG. 7.

THE three angles of a ſpherical triangle are greater than two right angles, and leſs than ſix right angles.

The meaſures of the angles A, B, C, in the triangle ABC, together with the three ſides of the ſupplemental triangle DEF, are (10. of this) equal to three ſemicircles; but the three ſides of the triangle FDE, are (7. of this) leſs than two ſemicircles;

micircles; therefore the meaſures of the angles A, B, C are reater than a ſemicircle; and hence the angles A, B, C are reater than two right angles.

All the external and internal angles of any triangle are equal ſix right angles; therefore, all the internal angles are leſs an ſix right angles.

PROP. XII. FIG. 8.

F from any point C, which is not the pole of the great circle ABD, there be drawn arches of great circles A, CD, CE, CF, &c. the greateſt of theſe is CA, hich paſſes thro' H the pole of ABD, and CB the reainder of ACB is the leaſt, and of any others CD, CE, F, &c. CD, which is nearer to CA, is greater than CE, hich is more remote.

Let the common ſection of the plains of the great circles CB, ADB be AB; and from C, draw CG perpendicular to B, which will alſo be perpendicular to the plain ADB; (4. Def. 11.) join GD, GE, GF, CD, CE, CF, CA, CB.

Of all the ſtraight lines drawn from G to the circumference ADB, GA is the greateſt, and GB the leaſt; (7. 3.) and GD hich is nearer to GA is greater than GE, which is more emote. The triangles CGA, CGD are right angled at G, nd they have the common ſide CG; therefore the ſquares of G, GA together, that is, the ſquare of CA, is greater than e ſquares of CG, GD together, that is, the ſquare of CD; nd CA is greater than CD, and therefore the arch CA is reater than CD. In the ſame manner, ſince GD is greater an GE, and GE than GF, &c. it is ſhewn that CD is reater than CE, and CE than CF, &c. and conſequently, the rch CD greater than the arch CE, and the arch CE greater an the arch CF, &c. And ſince GA is the greateſt, and GB e leaſt of all the ſtraight lines drawn from G to the circumrence ADB, it is manifeſt that CA is the greateſt, and CB e leaſt of all the ſtraight lines drawn from C to that circumrence; and therefore the arch CA is the greateſt, and CB the aſt of all the circles drawn through C, meeting ADB. Q. D.

PROP.

PROP. XIII. FIG. 9.

IN a right angled ſpherical triangle the ſides are of t
ſame affection with the oppoſite angles; that is, if t
ſides be greater or leſs than quadrants, the oppoſite angl
will be greater or leſs than right angles.

Let ABC be a ſpherical triangle right-angled at A, a
ſide AB, will be of the ſame affection with the oppoſite ang
ACB.

Caſe 1. Let AB be leſs than a quadrant, let AE be a qu
drant, and let EC be a great circle paſſing through E,
Since A is a right angle, and AE a quadrant, E is the p
of the great circle AC, and ECA a right angle; but ECA
greater than BCA, therefore BCA is leſs than a right ang
Q. E. D.

Fig. 10. Caſe 2. Let AB be greater than a quadrant, make AE a qu
drant, and let a great circle paſs thro' C, E. ECA is a right an
as before, and BCA is greater than ECA, that is, greater tha
right angle. Q. E. D.

PROP. XIV.

IF the two ſides of a right-angled ſpherical triangle be
the ſame affection, the hypothenuſe will be leſs tha
quadrant; and if they be of different affection, the h
pothenuſe will be greater than a quadrant.

Let ABC be a right-angled ſpherical triangle, if the two ſi
AB, AC be of the ſame or of different affection, the hypot
nuſe BC will be leſs or greater than a quadrant.

Fig. 9. Caſe 1ſt. Let AB, AC be each leſs than a quadrant.
AE, AG be quadrants; G will be the pole of AB, and E
pole of AC, and EC a quadrant; but, by Prop. 12. CE
greater than CB, ſince CB is farther off from CGD than
In the ſame manner, it is ſhewn that CB, in the triangle CE
where the two ſides CD, BD are each greater than a q
drant, is leſs than CE, that is, leſs than a quadrant.
E. D.

Caſe 2. Let AC be leſs, and AB greater than a quadrant; Fig. 10.
en the hypothenuſe BC will be greater than a quadrant; for
e AE be a quadrant, then E is the pole of AC, and EC will
a quadrant. But CB is greater than CE by Prop. 12. ſince
C paſſes thro' the pole of ABD. Q. E. D.

PROP. XV.

F the hypothenuſe of a right-angled triangle be greater
or leſs than a quadrant, the ſides will be of different
the ſame affection.

This is the converſe of the preceeding, and demonſtrated in
e ſame manner.

PROP. XVI.

N any ſpherical triangle ABC, if the perpendicular AD
from A upon the baſe BC fall within the triangle, the
gles B and C at the baſe will be of the ſame affection;
d if the perpendicular fall without the triangle, the
gles B and C will be of different affection.

1. Let AD fall within the triangle; then (13. of this) ſince Fig. 11.
DB, ADC are right-angled ſpherical triangles, the angles B,
muſt each be of the ſame affection as AD.

2. Let AD fall without the triangle, then (13. of this) the Fig. 12.
gle B is of the ſame affection as AD; and by the ſame, the
gle ACD is of the ſame affection as AD; therefore the angle
CB and AD are of different affection, and the angles B and
CB of different affection.

Cor. Hence if the angles B and C be of the ſame affection,
e perpendicular will fall within the baſe; for, if it did not,
6. of this) B and C would be of different affection. And if
e angles B and C be of oppoſite affection, the perpendicular
ll fall without the triangle; for, if it did not, (16 of this) the
gles B and C would be of the ſame affection, contrary to the
ppoſition.

PROP.

PROP. XVII. FIG. 13.

IN right-angled ſpherical triangles, the ſine of either the ſides about the right angle, is to the radius the ſphere, as the tangent of the remaining ſide is to tangent of the angle oppoſite to that ſide.

Let ABC be a triangle, having the right angle at A; let AB be either of the ſides, the ſine of the ſide AB will to the radius, as the tangent of the other ſide AC to the t gent of the angle ABC, oppoſite to AC. Let D be the c ter of the ſphere; join AD, BD, CD, and let AE be dra perpendicular to BD, which therefore will be the ſine of arch AB and from the point E, let there be drawn in plane BDC the ſtraight line EF at right angles to BD, meet DC in F, and let AF be joined. Since therefore the ſtrai line DE is at right angles to both EA and EF, it will alſo at right angles to the plane AEF, (4. 11.) wherefore the pl ABD, which paſſes thro' DE, is perpendicular to the pl AEF, (18. 11.) and the plane AEF perpendicular to ABD: T plane ACD or AFD is alſo perpendicular to the ſame AB Therefore the common ſection, viz. the ſtraight line AF, i right angles to the plane ABD: (19. 11.) And FAE, FAD right angles; (3. Def. 11.) therefore AF is the tangent the arch AC; and in the rectilineal triangle AEF, havin right angle at A, AE will be to the radius as AF to the t gent of the angle AEF, (1. Pl. Tr.); but AE is the ſine the arch AB, and AF the tangent of the arch AC, and angle AEF is the inclination of the planes CBD, ABD, Def. 11.) or the ſpherical angle ABC: Therefore the ſine of arch AB is to the radius as the tangent of the arch AC, the tangent of the oppoſite angle ABC.

COR. 1. If therefore of the two ſides, and an angle opp ſite to one of them, any two be given, the third will alſo given.

COR. 2. And ſince by this propoſition the ſine of the ſi AB is to the radius, as the tangent of the other ſide A

to the tangent of the angle ABC oppoſite to that ſide; and as the radius is to the co-tangent of the angle ABC, ſo is the tangent of the ſame angle ABC to the radius, (Cor. 2. Def. Pl. Tr.) by equality, the ſine of the ſide AB is to the co-tangent of the angle ABC adjacent to it, as the tangent of the other ſide AC to the radius.

PROP. XVIII. FIG. 13.

IN right angled ſpherical triangles the ſine of the hypothenuſe is to the radius, as the ſine of either ſide is to the ſine of the angle oppoſite to that ſide.

Let the triangle ABC be right-angled at A, and let AC be either of the ſides; the ſine of the hypothenuſe BC will be to the radius as the ſine of the arch AC is to the ſine of the angle ABC.

Let D be the center of the ſphere, and let CG be drawn perpendicular to DB, which will therefore be the ſine of the hypothenuſe BC; and from the point G let there be drawn in the plane ABD the ſtraight line GH perpendicular to DB, and let CH be joined: CH will be at right angles to the plane ABD, as was ſhewn in the preceeding Propoſition of the ſtraight line FA: Wherefore CHD, CHG are right angles, and CH is the ſine of the arch AC; and in the triangle CHG, having the right angle CHG, CG is to the radius as CH to the ſine of the angle CGH: (1. Pl. Tr.) But ſince CG, HG are at right angles to DGB, which is the common ſection of the planes CBD, ABD, the angle CGH will be equal to the inclination of theſe planes; (6. Def. 11.) that is, to the ſpherical angle ABC. The ſine, therefore, of the hypothenuſe CB is to the radius as the ſine of the ſide AC is to the ſine of the oppoſite angle ABC. Q. E. D.

COR. Of theſe three, viz. the hypothenuſe, a ſide, and the angle oppoſite to that ſide, any two being given, the third is alſo given, by Prop. 2.

PROP.

PROP. XIX. FIG. 14.

IN right-angled ſpherical triangles, the co-ſine of th hypothenuſe is to the radius as the co-tangent of e ther of the angles is to the tangent of the remainin angle.

Let ABC be a ſpherical triangle, having a right angle A, the co-ſine of the hypothenuſe BC will be to the radi as the co-tangent of the angle ABC to the tangent of t angle ACB.

Deſcribe the circle DE, of which B is the pole, and it meet AC in F, and the circle BC in E; and ſince the circ BD paſſes thro' the pole B of the circle DF, DF will alſo pa thro' the pole of BD. (13. 18. 1. Theod. ſph.) And ſince AC perpendicular to BD, AC will alſo paſs thro' the pole BD; wherefore the pole of the circle BD will be found in t point where the circles AC, DE meet, that is, in the point F The arches FA, FD are therefore quadrants, and likewiſe t arches BD, BE: In the triangle CEF, right-angled at the poi E, CE is the complement of the hypothenuſe BC of the tr angle ABC, EF is the complement of the arch ED, which the meaſure of the angle ABC, and FC the hypothenuſe the triangle CEF, is the complement of AC, and the ar AD, which is the meaſure of the angle CFE, is the compl ment of AB.

But (17. of this) in the triangle CEF, the ſine of the ſi CE is to the radius, as the tangent of the other ſide is to t tangent of the angle ECF oppoſite to it, that is, in the triang ABC, the co-ſine of the hypothenuſe BC is to the radius, the co-tangent of the angle ABC is to the tangent of the ang ACB. Q. E. D.

COR. 1. Of theſe three viz. the hypothenuſe and the tw angles, any two being given, the third will alſo be given.

COR. 2. And ſince by this Propoſition the co-ſine of th hypothenuſe BC is to the radius as the co-tangent of th angle ABC to the tangent of the angle ACB But as the ra dius is to the co-tangent of the angle ACB, ſo is the tange of the ſame to the radius; (Cor. 2. Def. Pl. Tr.) and, e æquo, the co-ſine of the hypothenuſe BC is to the co-tange

the angle ACB, as the co-tangent of the angle ABC to the adius.

PROP. XX. FIG. 14.

IN right angled ſpherical triangles, the co-ſine of an angle is to the radius, as the tangent of the ſide adcent to that angle is to the tangent of the hypothenuſe.

The ſame conſtruction remaining; in the triangle CEF, 7. of this) the ſine of the ſide EF is to the radius, as the ngent of the other ſide CE is to the tangent of the angle FE oppoſite to it; that is, in the triangle ABC, the co-ſine the angle ABC is to the radius as (the co-tangent of the ypothenuſe BC to the co-tangent of the ſide AB, adjacent to ABC or as) the tangent of the ſide AB to the tangent of the ypothenuſe, ſince the tangents of two arches are reciproally proportional to their co-tangents. (Cor. 1. Def. Pl. f.)

COR. And, ſince by this propoſition the co-ſine of the ngle ABC is to the radius, as the tangent of the ſide AB is the tangent of the hypothenuſe BC; and as the radius is to e co-tangent of BC, ſo is the tangent of BC to the radius; y equality, the co-ſine of the angle ABC will be to the o-tangent of the hypothenuſe BC, as the tangent of the ſide A djacent to the angle ABC to the radius.

PROP. XXI. FIG. 14.

IN right angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the hypothenuſe is to the co-ſine of the other ſide.

The ſame conſtruction remaining; in the triangle CEF, the ine of the hypothenuſe CF is to the radius, as the ſine of the ide CE to the ſine of the oppoſite angle CFE; (18. of this) hat is, in the triangle ABC the co-ſine of the ſide CA is to he radius as the co-ſine of the hypothenuſe BC to the coine of the other ſide BA. Q. E. D.

PROP. XXII. FIG. 14.

IN right angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the angle oppoſite to that ſide is to the ſine of the other angle.

The ſame conſtruction remaining; in the triangle CEF the ſine of the hypothenuſe CF is to the radius as the ſine of the ſide EF is to the ſine of the angle ECF oppoſite to it; that is, in the triangle ABC, the co-ſine of the ſide CA is to the radius, as the co-ſine of the angle ABC oppoſite to it, is to the ſine of the other angle. Q. E. D.

Of the CIRCULAR PARTS.

IN any right angled ſpherical triangle ABC, the complement of the hypothenuſe, the complements of the angles, and the two ſides, are called *The circular parts of the triangle*, as if it were following each other in a circular order, from whatever part we begin: Thus, if we begin at the complement of the hypothenuſe, and proceed towards the ſide BA, the parts following in order will be the complement of the hypothenuſe, the complement of the angle B, the ſide BA the ſide AC, (for the right angle at A is not reckoned among the parts,) and laſtly, the complement of the angle C. And thus at whatever part we begin, if any three of theſe five be taken, they either will be all contiguous or adjacent, or one of them will not be contiguous to either of the other two: In the firſt caſe, the part which is between the other two is called the *Middle part*, and the other two are called *Adjacent extremes*. In the ſecond caſe, the part which is not contiguous to either of the other two is called the *Middle part*, and the other two *Oppoſite extremes*. For example, if the three parts be the complement of the hypothenuſe BC, the complement of the angle B, and the ſide BA; ſince theſe three are contiguous to each other, the complement of the angle B will be the middle part, and the complement of the hypothenuſe BC and the ſide BA will be adjacent extremes: But if the complement of the hypothenuſe BC, and the ſides BA, AC be taken; ſince the complement of the hypothenuſe is not adjacent to either of the ſides, viz. on account of the complements of the two angles B and C interveening between it and the ſides, the complement of the hypothenuſe BC will be the middle part, and the ſides, BA, AC oppoſite extremes. The moſt acute and ingenious Baron Napier, the inventor of Logarithms, contrived the two following rules concerning theſe parts, by means of which all the caſes of right-angled ſpherical triangles are reſolved with the greateſt eaſe. Fig. 15.

RULE I.

The rectangle contained by the radius and the ſine of the

middle part, is equal to the rectangle contained by the tangents of the adjacent parts.

RULE II.

The rectangle contained by the radius, and the ſine of the middle part is equal to the rectangle contained by the coſines of the oppoſite parts.

Theſe rules are demonſtrated in the following manner.

Fig. 16. Firſt, Let either of the ſides, as BA, be the middle part, and therefore the complement of the angle B, and the ſide AC will be adjacent extremes. And by Cor. 2. Prop. 17. of this, S, BA is to the Co-T, B, as T, AC is to the radius, and therefore R × S, BA = Co-T, B × T, AC.

The ſame ſide BA being the middle part, the complement of the hypothenuſe, and the complement of the angle C, are oppoſite extremes; and by Prop. 18. S, BC is to the radius as S, BA to S, C; therefore R × S, BA = S, BC × S, C.

Secondly, Let the complement of one of the angles, as B, be the middle part, and the complement of the hypothenuſe, and the ſide BA will be adjacent extremes: And by Cor. Prop. 20. Co S, B is to Co-T, BC, as T, BA is to the radius, and therefore R × Co S, B = Co-T, BC × T, BA.

Again, Let the complement of the angle B be the middle part, and the complement of the angle C, and the ſide AC will be oppoſite extremes: And by Prop. 22. Co S, AC is to the radius, as Co S, B is to S, C: And therefore R × Co S, B = Co S, AC × S, C.

Thirdly, Let the complement of the hypothenuſe be the middle part, and the complements of the angles B, C, will be adjacent extremes: But by Cor. 2. Prop. 19. Co S, BC is to Co-T, C as to Co-T, B to the radius: Therefore R × Co S, BC = Co-T, B × Co-T, C.

Again, Let the complement of the hypothenuſe be the middle part, and the ſides AB, AC will be oppoſite extremes. But by Prop. 21. Co S, AC is to the radius, Co S, BC to Co S, BA; therefore R × Co S, BC = Co S, BA × Co S, AC. Q. E. D.

SOLU

SOLUTION of the Sixteen CASES of right-angled Spherical triangles. Fig. 16.

GENERAL PROPOSITION.

IN a right angled ſpherical triangle, of the three ſides, and three angles, any two being given beſides the right angle, the other three may be found.

In the following table the ſolutions are derived from the preceeding Propoſitions. It is obvious that the ſame ſolutions may be derived from Baron Napier's two rules above demonſtrated, which, as they are eaſily remembered, are commonly uſed in practice.

Caſe	Given	Sought	
1	AC, C	B	R : Co S, AC : : S, C : Co S, B : And B is of the ſame ſpecies with CA, by 22. and 13.
2	AC, B	C	Co S, AC : R : : Co S, B : S, C : By 22.
3	B, C	AC	S, C : Co S, B : : R : Co S, AC : By 22. and AC is of the ſame ſpecies with B. 13.
4	BA, AC	BC	R : CoS, BA : : CoS, AC : CoS, BC 21. and if both BA, AC be greater or leſs than a quadrant, BC will be leſs than a quadrant. But if they be of different affection, BC will be greater than a quadrant. 14.
5	BA, BC	AC	Co S, BA : R : : CoS, BC : CoS, AC 21. and if BC be greater or leſs than a quadrant, BA, AC will be of different or the ſame affection : By 15.
6	BA, AC	B	S, BA : R : : T, CA : T, B. 17. and B is of the ſame affection with AC, 13.

Cafe	Given	Sought	
7	BA, B	AC	R : S, BA : : T, B: T, AC. 17. And AC i of the ſame affection with B. 13.
8	AC, B	BA	T, B : R : : T, CA : S, BA. 17.
9	BC, C	AC	R : Co S, C : : T, BC: T, CA. 20. If BC b leſs or greater than a quadrant, C and B will be of the ſame or different affection. 15 13.
10	AC, C	BC	Co S, C : R : : T, AC: T, BC . 20. And BC i leſs or greater than a quadrant, according a C and AC or C and B are of the ſame o different affections. 14. 1.
11	BC, CA	C	T, BC : R : : T, CA : Co S, C. 20. If BC be leſs or greater than a quadrant, C and AB, and therefore CA and C, are of the ſame or different affection. 15.
12	BC, B	AC	R : S, BC : : S, B : S, AC. 18. And AC is o the ſame affection with B.
13	AC, B	BC	S, B : S, AC : : R : S, BC : 18.
14	BC, AC	B	S, BC : R : : S, AC : S, B: 18. And B is o the ſame affection with AC.
15	B, C	BC	T, C : R : : Co T, B : Co S, BC. 19. And according as the angles B and C are of dif ferent or the ſame affection, BC will be greater or leſs than a quadrant. 14.
16	BC, C	B	R : Co S, BC : : T, C : Co T, B. 19. If BC be leſs or greater than a quadrant, C and B will be of the ſame or different affection. 15

The ſecond, eighth, and thirteenth caſes, which are commonly called ambiguous, admit of two ſolutions: For in theſe it is not determined whether the ſide or meaſure of the angle ſought be greater or leſs than a quadrant.

PROP. XXIII. FIG. 16.

IN ſpherical triangles, whether right-angled or oblique angled, the ſines of the ſides are proportional to the ſines of the angles oppoſite to them.

Firſt, Let ABC be a right-angled triangle, having a right angle at A; therefore by Prop 18. the ſine of the hypotheneuſe BC is to the radius (or the ſine of the right angle at A) as the ſine of the ſide AC to the ſine of the angle B. And, in like manner, the ſine of BC is to the ſine of the angle A, as the ſine of AB to the ſine of the angle C; wherefore (11. 5.) the ſine of the ſide AC is to the ſine of the angle B, as the ſine of AB to the ſine of the angle C.

Secondly, Let BCD be an oblique-angled triangle, the ſine of either of the ſides BC, will be to the ſine of either of the other two CD, as the ſine of the angle D oppoſite to BC is to the ſine of the angle B oppoſite to the ſide CD. Through the point C, let there be drawn an arch of a great circle CA perpendicular upon BD; and in the right-angled triangle ABC (18. of this) the ſine of BC is to the radius, as the ſine of AC to the ſine of the angle B; and in the triangle ADC (by 18. of this:) And, by inverſion, the radius is to the ſine of DC as the ſine of the angle D to the ſine of AC: Therefore, ex aequo perturbate, the ſine of BC is to the ſine of DC, as the ſine of the angle D to the ſine of the angle B. Q. E. D. Fig. 17. 18

PROP. XXIV. FIG. 17. 18.

IN oblique-angled ſpherical triangles, having drawn a perpendicular arch from any of the angles upon the oppoſite ſide, the co-ſines of the angles at the baſe are proportional to the ſines of the vertical angles.

Let BCD be a triangle, and the arch CA perpendicular to the baſe BD; the co-ſine of the angle B will be to the co-ſine of the angle D, as the ſine of the angle BCA to the ſine of the angle DCA.

For by 22. the co-ſine of the angle B is to the ſine of the angle BCA as (the co-ſine of the ſide AC is to the radius that is, by Prop. 22. as) the co-ſine of the angle D to the ſine of the angle DCA; and, by permutation, the co-ſine of the angle B is to the co-ſine of the angle D, as the ſine of the angle BCA to the ſine of the angle DCA. Q. E. D.

PROP. XXV. FIG. 17. 18.

THE ſame things remaining, the co-ſines of the ſides BC, CD, are proportional to the co-ſines of the baſes BA, AD.

For by 21. the co-ſine of BC is to the co-ſine of BA as (the co-ſine of AC to the radius; that is, by 21. as) the co-ſine of CD is to the co-ſine of AD: Wherefore, by permutation, the co-ſines of the ſides BC, CD are proportional to the co-ſines of the baſes BA, AD. Q. E. D.

PROP. XXVI. FIG. 17. 18.

THE ſame conſtruction remaining, the ſines of the baſes BA, AD are reciprocally proportional to the tangents of the angles B and D at the baſe.

For by 17. the ſine of BA is to the radius, as the tangent of AC to the tangent of the angle B; and by 17. and inverſion, the radius is to the ſine of AD, as the tangent of D to the tangent of AC: Therefore, ex aequo perturbate, the ſine of BA is to the ſine of AD, as the tangent of D to the tangent of B.

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PROP. XXVII. FIG. 17. 18.

THE co-ſines of the vertical angles are reciprocally proportional to the tangents of the ſides.

For by Prop. 20. the co-ſine of the angle BCA, is to the ...dius as the tangent of CA is to the tangent of BC; and by ...e ſame Prop. 20. and by inverſion, the radius is to the co-...e of the angle DCA, as the tangent of DC to the tangent ...CA: Therefore, ex aequo perturbate, the co-ſine of the ...gle BCA is to the co-ſine of the angle DCA, as the tangent ...DC is to the tangent of BC. Q. E. D.

LEMMA. FIG. 19. 20.

IN right-angled plain triangles, the hypothenuſe is to ... the radius, as the exceſs of the hypothenuſe above ...ither of the ſides to the verſed ſine of the acute angle ...djacent to that ſide, or as the ſum of the hypothenuſe, ...nd either of the ſides to the verſed ſine of the exterior ...ngle of the triangle.

Let the triangle ABC have a right angle at B; AC will ...e to the radius as the exceſs of AC above AB, to the verſed ...ne of the angle A adjacent to AB; or as the ſum of AC, ...AB to the verſed ſine of the exterior angle CAK.

With any radius DE, let a circle be deſcribed, and from ...the center let DF be drawn to the circumference, making ...he angle EDF equal to the angle BAC, and from the point ...F, let FG be drawn perpendicular to DE: Let AH, AK be ...made equal to AC, and DL to DE: DG therefore is the co-...ine of the angle EDF or BAC, and GE its verſed ſine: And ...becauſe of the equiangular triangles ACB, DFG, AC or AH ...is to DF or DE, as AB to DG: Therefore (19. 5.) AC is to ...he radius DE as BH to GE, the verſed ſine of the angle ...EDF or BAC: And ſince AH is to DE, as AB to DG, (12. ...5.) AH or AC will be to the radius DE as KB to LG, the ...verſed ſine of the angle LDF or KAC. Q. E. D.

PROP.

PROP. XXVIII. FIG. 21. 22.

IN any ſpherical triangle, the rectangle contained the ſines of two ſides, is to the ſquare of the radius, the exceſs of the verſed ſines of the third ſide or ba and the arch, which is the exceſs of the ſides, is to t verſed ſine of the angle oppoſite to the baſe.

Let ABC be a ſpherical triangle, the rectangle contain by the ſines of AB, BC will be to the ſquare of the radi as the exceſs of the verſed ſines of the baſe AC, and of t arch, which is the exceſs of AB, BC to the verſed ſine the angle ABC oppoſite to the baſe.

Let D be the center of the ſphere, and let AD, BD, C be joined, and let the ſines AE, CF, CG of the arches A BC, AC be drawn; let the ſide BC be greater than BA, a let BH be made equal to BC; AH will therefore be the e ceſs of the ſides BC, BA; let HK be drawn perpendicular AD, and ſince AG is the verſed ſine of the baſe AC, a AK the verſed ſine of the arch AH, KG is the exceſs of t verſed ſines of the baſe AC, and of the arch AH, which the exceſs of the ſides BC, BA: Let GL likewiſe be dra parallel to KH, and let it meet FH in L, let CL, DH joined, and let AD, FH meet each other in M.

Since therefore in the triangles CDF, HDF, DC, D are equal, DF is common, and the angle FDC equal to t angle FDH, becauſe of the equal arches BC, BH, the ba HF will be equal to the baſe FC, and the angle HFD equ to the right angle CFD: The ſtraight line DF therefore (11.) is at right angles to the plane CFH: Wherefore the pla CFH is at right angles to the plane BDH, which paſſes th DF. (18. 11.) In like manner, ſince DG is at right angles both GC and GL, DG will be perpendicular to the pla CGL; therefore the plane CGL is at right angles to the pla BDH, which paſſes thro' DG: And it was ſhewn that t plane CFH or CFL, was perpendicular to the ſame pla BDH; therefore the common ſection of the planes CF CGL, viz. the ſtraight line GL, is perpendicular to the pla BDA, (19. 11.) and therefore CLF is a right angle: In t triangle CFL having the right angles CLF, by the Lemma

the radius as LH, the exceſs, viz. of CF or FH above is to the verſed ſine of the angle CFL; but the angle L is the inclination of the planes BCD, BAD, ſince FC, are drawn in them at right angles to the common ſec- BF: The ſpherical angle ABC is therefore the ſame the angle CFL; and therefore, CF is to the radius as to the verſed ſine of the ſpherical angle ABC; and ſince triangle AED is equiangular (to the triangle MFD, and refore) to the triangle MGL, AE will be to the radius of ſphere AD, (as MG to ML; that is, becauſe of the pa- els as) GK to LH: The ratio therefore which is compound- of the ratios of AE to the radius, and of CF to the ſame ius; that is, (23. 6.) the ratio of the rectangle contained AE, CF to the ſquare of the radius, is the ſame with the io compounded of the ratio of GK to LH, and the ratio LH to the verſed ſine of the angle ABC; that is, the ſame h the ratio of GK to the verſed ſine of the angle ABC; refore, the rectangle contained by AE, CF, the ſines of the es AB, BC, is to the ſquare of the radius as GK, the exceſs the verſed ſines AG, AK, of the baſe AC, and the arch H, which is the exceſs of the ſides to the verſed ſine of the gle ABC oppoſite to the baſe AC. Q. E. D.

PROP. XXIX. Fig. 23.

THE rectangle contained by half of the radius, and the exceſs of the verſed ſines of two arches, is e- al to the rectangle contained by the ſines of half the m, and half the difference of the ſame arches.

Let AB, AC be any two arches, and let AD be made e- al to AC the leſs; the arch DB therefore is the ſum, and arch CB the difference of AC, AB: Thro' E the center the circle, let there be drawn a diameter DEF, and AE ined, and CD likewiſe perpendicular to it in G; and let H be perpendicular to AE, and AH will be the verſed ſine the arch AB, and AG the verſed ſine of AC, and HG the ceſs of theſe verſed ſines: Let BD, BC, BF be joined, and alſo meeting BH in K.

Since therefore BH, CG are parallel, the alternate angles C, KCG will be equal; but KCG is in a ſemicircle, and therefore

therefore a right angle; therefore BKC is a right angle; in the triangles DFB, CBK, the angles FDB, BCK in ſame ſegment are equal, and FBD, BKC are right angl the triangles DFB, CBK are therefore equiangular; where DF is to DB, as BC to CK, or HG; and therefore the re angle contained by the diameter DF, and HG is equal to t contained by DB, BC; wherefore the rectangle contained a fourth part of the diameter, and HG, is equal to that c tained by the halves of DB, BC: But half the chord DB is ſine of half the arch DAB, that is, half the ſum of the arc AB, AC; and half the chord of BC is the ſine of half arch BC, which is the difference of AB, AC. Whence Propoſition is manifeſt.

PROP. XXX. FIG. 19. 24.

THE rectangle contained by half of the radius, a the verſed ſine of any arch, is equal to the ſqu of the ſine of half the ſame arch.

Let AB be an arch of a circle, C its center, and AC, C BA being joined: Let AB be biſected in D, and let CD joined, which will be perpendicular to BA, and biſect it in (4. 1.) BE or AE therefore is the ſine of the arch DB AD, the half of AB: Let BF be perpendicular to AC, a AF will be the verſed ſine of the arch BA; but, becauſe of ſimilar triangles CAE, BAF, CA is to AE as AB, that twice AE to AF; and by halving the antecedents, half of radius CA is to AE the ſine of the arch AD, as the ſame AE AF the verſed ſine of the arch AB. Wherefore by 16. 6. Propoſition is manifeſt.

PROP. XXXI. FIG. 25.

IN a ſpherical triangle, the rectangle contained by ſines of the two ſides, is to the ſquare of the radi as the rectangle contained by the ſine of the arch wh is half the ſum of the baſe, and the exceſs of the ſid and the ſine of the arch, which is half the difference the ſame to the ſquare of the ſine of half the angle poſite to the baſe.

et ABC be a ſpherical triangle, of which the two ſides are
BC, and baſe AC, and let the leſs ſide BA be produced, ſo
BD ſhall be equal to BC: AD therefore is the exceſs of
BA; and it is to be ſhewn, that the rectangle contained
he ſines of BC, BA is to the ſquare of the radius, as the
angle contained by the ſine of half the ſum of AC, AD,
the ſine of half the difference of the ſame AC, AD to the
are of the ſine of half the angle ABC, oppoſite to the baſe

ince by Prop. 28. the rectangle contained by the ſines of
ſides BC, BA is to the ſquare of the radius, as the exceſs
he verſed ſines of the baſe AC and AD, to the verſed ſine
the angle B; that is, (1. 6.) as the rectangle contained by
the radius, and that exceſs, to the rectangle contained by
the radius, and the verſed ſine of B; therefore (29. 30. of
,) the rectangle contained by the ſines of the ſides BC,
is to the ſquare of the radius, as the rectangle contained
he ſine of the arch, which is half the ſum of AC, AD,
the ſine of the arch which is half the difference of the
e AC, AD is to the ſquare of the ſine of half the angle
C. Q. E. D.

SOLU

SOLUTION of the twelve CASES of oblique angled SPHERICAL TRIANGLES.

GENERAL PROPOSITION.

IN any oblique angled ſpherical triangle, of the ſix ſides and three angles, any three being given, the other three may be found.

Fig. 26. 27.

	GIVEN.	SOUGHT.	
1	B, D, and BC, two angles and a ſide oppoſite one of them.	C.	Co S, BC : R : : Co T, B : T, BC 19. Likewiſe by 24. Co S, B : S, BCA : Co S, D : S, DCA; wherefore BCD the ſum or difference of the angles DC BCA according as the perpendicular falls within or without the trian BCD; that is, (16. of this,) accordi as the angles B, D are of the ſame different affection.
2	B, C, and BC two angles and the ſide between them.	D	Co S, BC : R : : Co T, B : T, BCA 19. and alſo by 24. S, BCA : S, DCA : Co S, B : Co S, D; and according as angle BCA is leſs or greater than BC the perpendicular CA falls within without the triangle BCD; and the fore (16. of this,) the angles B, D w be of the ſame or different affection.
3	BC, CD, and B.	BD.	R : Co S, B : : T, BC : T, BA. and Co S, BC : Co S, BA : : Co S, DC Co S, DA. 25. and BD is the ſum difference of BA, DA.
4	BC, DB and B.	CD.	R : Co S, B : : T, BC : T, BA. and Co S, BA : Co S, BC : : Co S, DA Co S, DC. 25. and according as DA, are of the ſame or different affection, will be leſs or greater than a quadra 14.

GI

Given.	Sought.	
B, D and BC.	DB.	R : Co S, B : : T, BC : T, BA. 20. and T, D : T, B : : S, BA : S, DA. 26. and BD is the sum or difference of BA, DA.
BC, BD and B.	D.	R : Co S, B : : T, BC ; T, BA. 20. and S, DA : S, BA : : T, B : T, D ; and according as BD is greater or less than BA, the angles B, D are of the same or different affection. 16.
BC, DC and B.	C.	Co S, BC : R : : Co T, B : T, BCA. 19. and T, DC : T, BC : : Co S, BCA : Co S, DCA 27. the sum or difference of the angles BCA, DCA is equal to the angle BCD.
B, C and BC.	DC.	Co S, BC : R : : Co T, B : T, BCA. 19. also by 27. Co S, DCA : Co S, BCA : : T, BC : T, DC 27. if DCA and B be of the same affection ; that is, (13.) if AD and CA be similar, DC will be less than a quadrant, 14. and if AD, CA be not of the same affection, DC is greater than a quadrant. 14.
BC, DC and B.	D.	S, CD : S, B : : S, BC : S, D.
B, D and BC.	DC.	S, D : S, BC : : S, B : S, DC.
BC, BA, AC. Fig. 25.	B.	S, AB × S, BC : Rq : : S, $\frac{AC+AD}{2}$ × S, $\frac{AC-AD}{2}$: Sq, $\frac{ABC}{2}$. See Fig. 25. AD being the difference of the sides BC, BA.

	Given.	Sought.	
			See Fig. 7.
12	A, B, C. Fig. 7.	The sides.	In the triangle DEF, DE, EF, F respectively the supplements of the sures of the given angles B, A, C i triangle BAC; the sides of the tri DEF are therefore given, and by th ceeding case the angles D, E, F m found, and the sides BC, BA, AC a supplements of the measures of angles.

The 3d, 5th, 7th, 9th, 10th, cases, which are com called ambiguous, admit of two solutions, either of whic answer the conditions required; for, in these cases, the sure of the angle or side sought, may be either greater than a quadrant, and the two solutions will be suppleme each other. (Cor. to Def. 4. 6. Pl. Tr.)

If from any of the angles of an oblique-angled spheric angle, a perpendicular arch be drawn upon the opposit most of the cases of oblique-angled triangles may be re by means of Napier's rules.

FINIS

16 OC 65

SPHERICAL TRIGONO

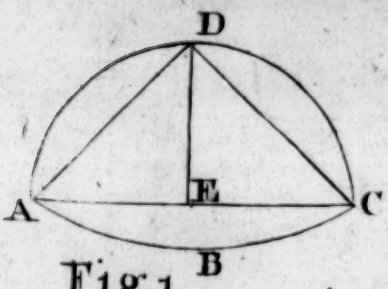
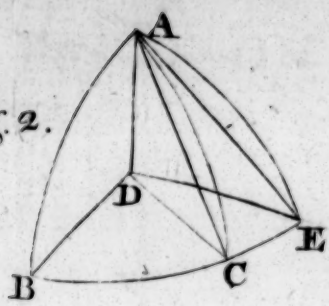
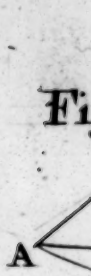


Fig. 1.

Fig. 2.





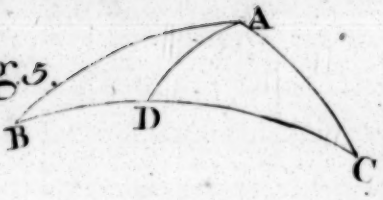


Fig. 5.

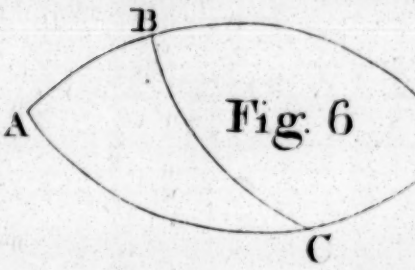


Fig. 6

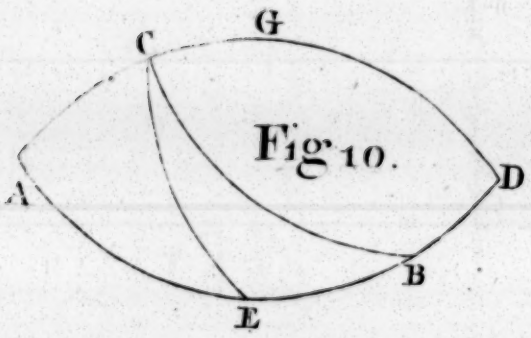


Fig. 10.

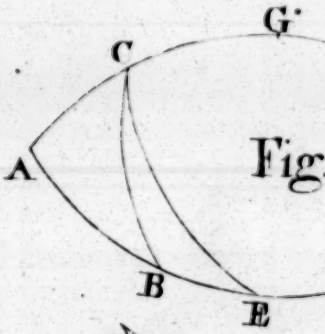


Fig.

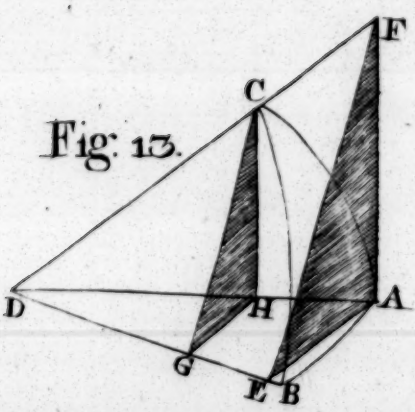


Fig. 13.

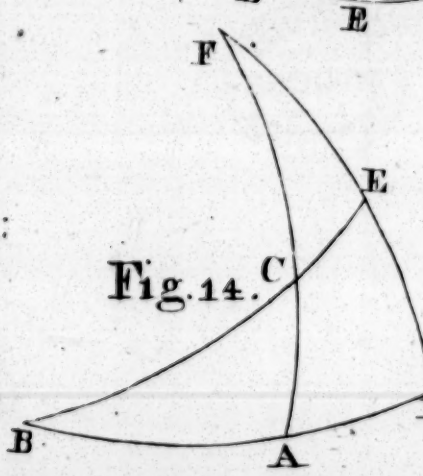
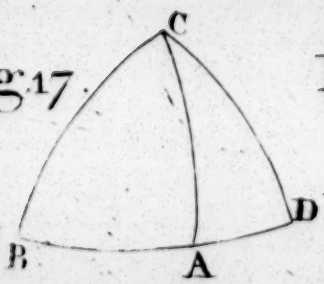


Fig. 14.



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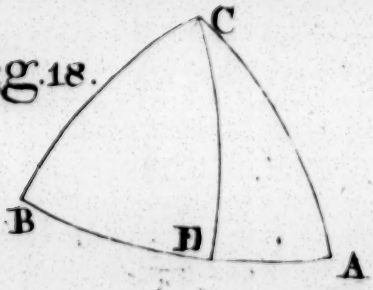

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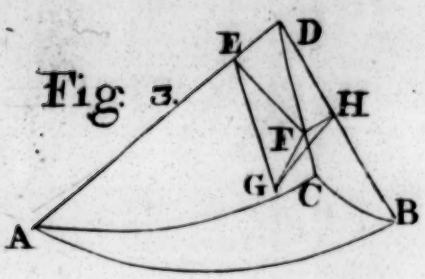

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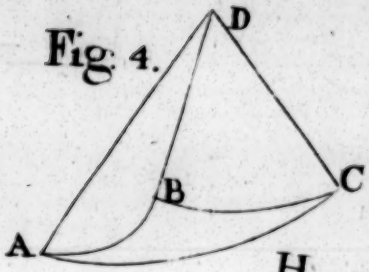

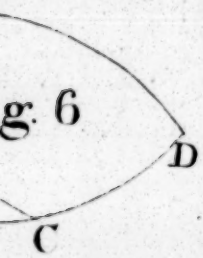

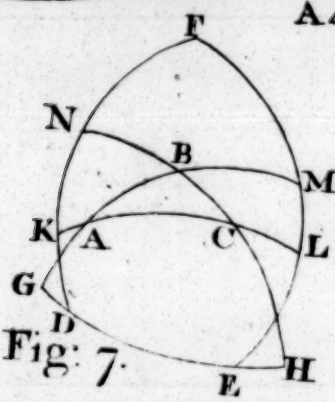

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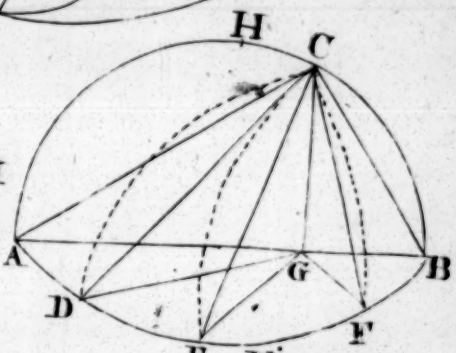

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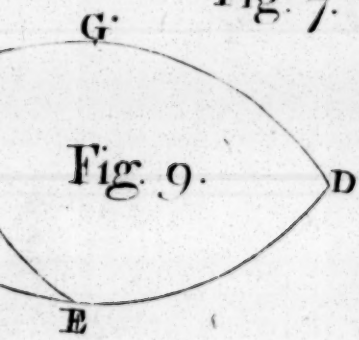

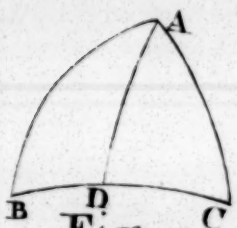

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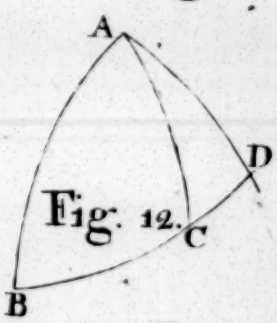

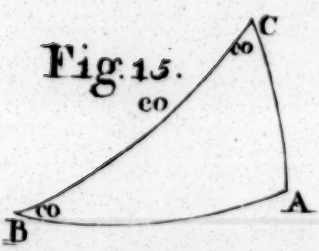

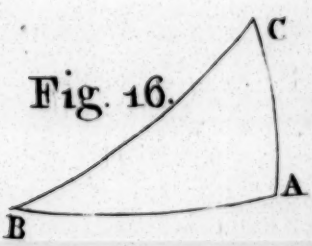

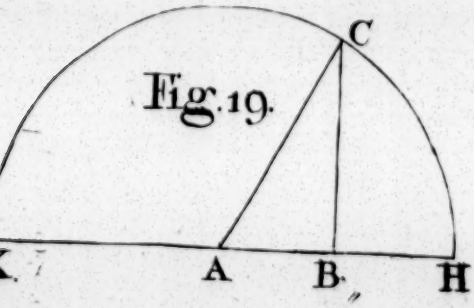

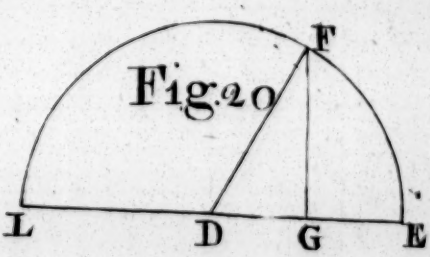

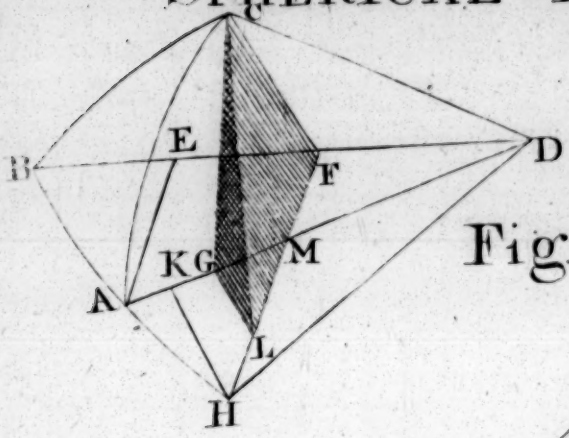

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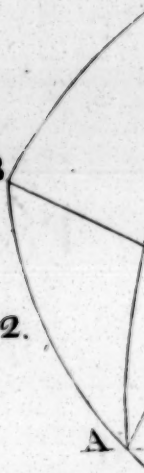

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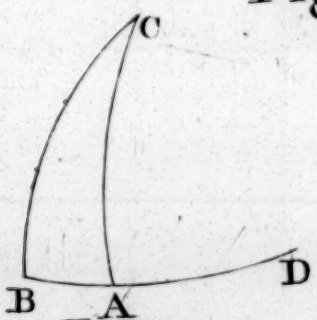

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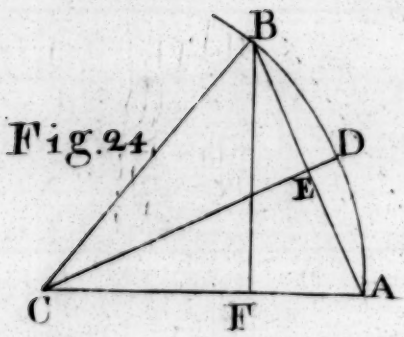

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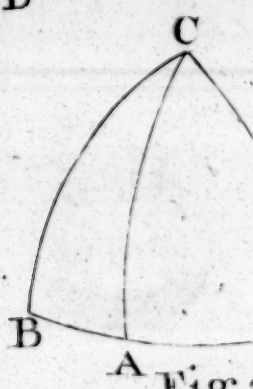

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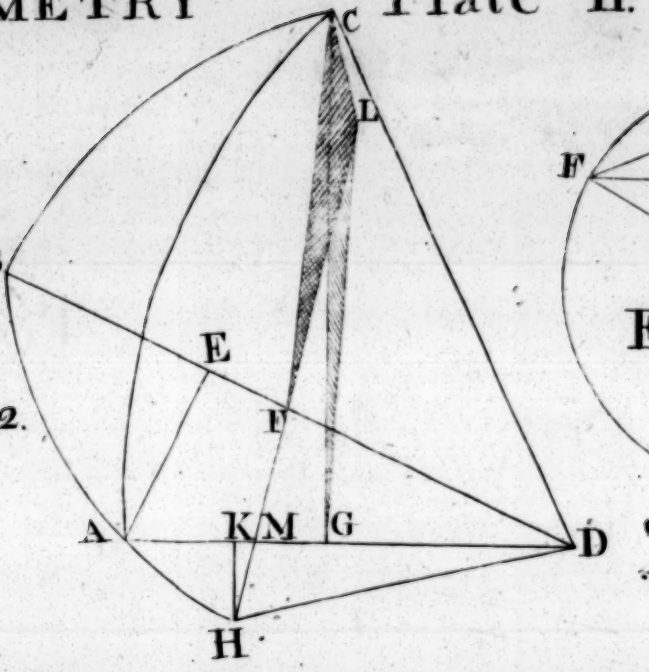

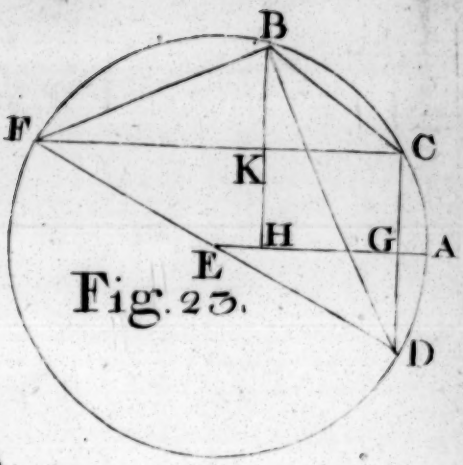

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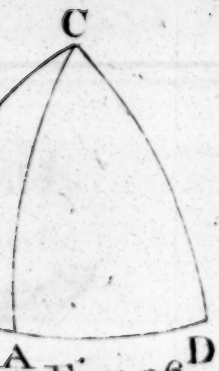
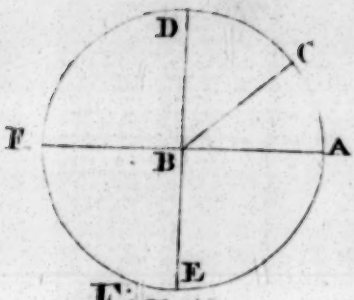


Fig. 26.

Fig. 27

PLANE



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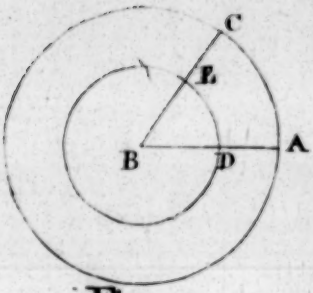

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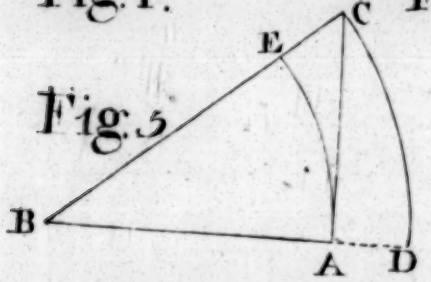

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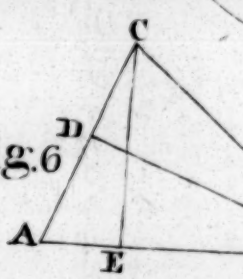

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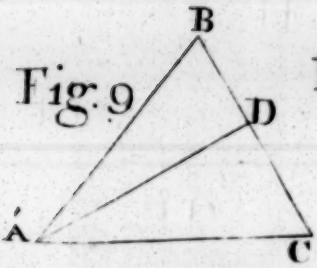

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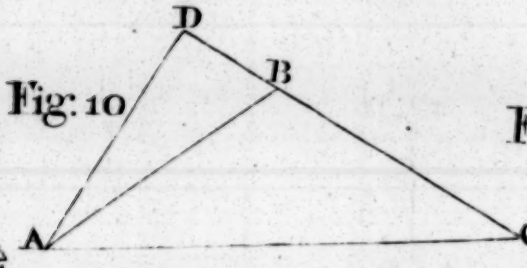

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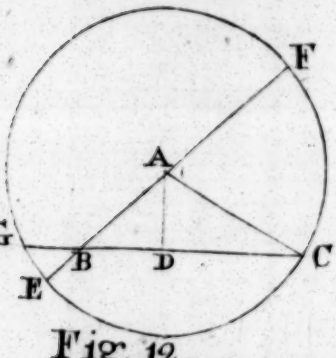

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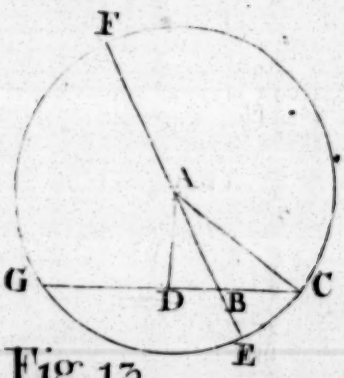

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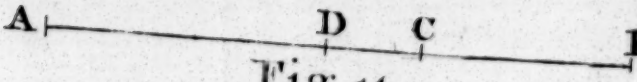

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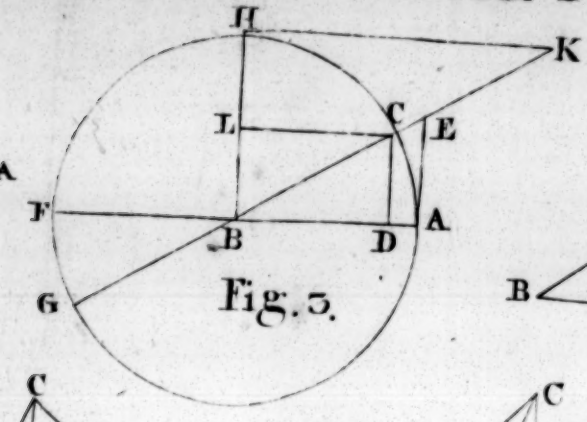

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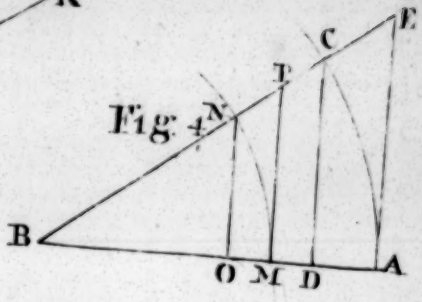

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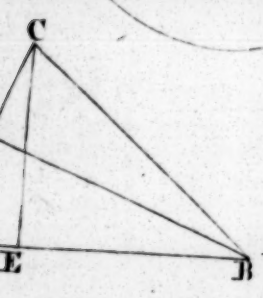

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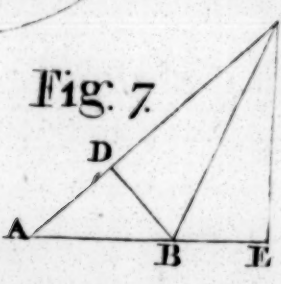

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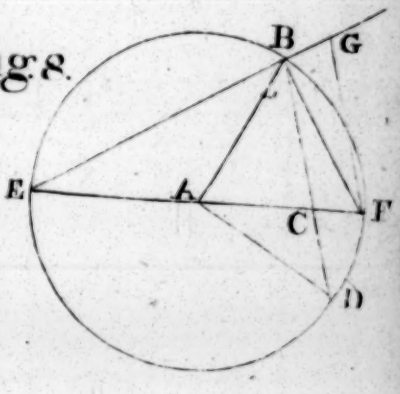

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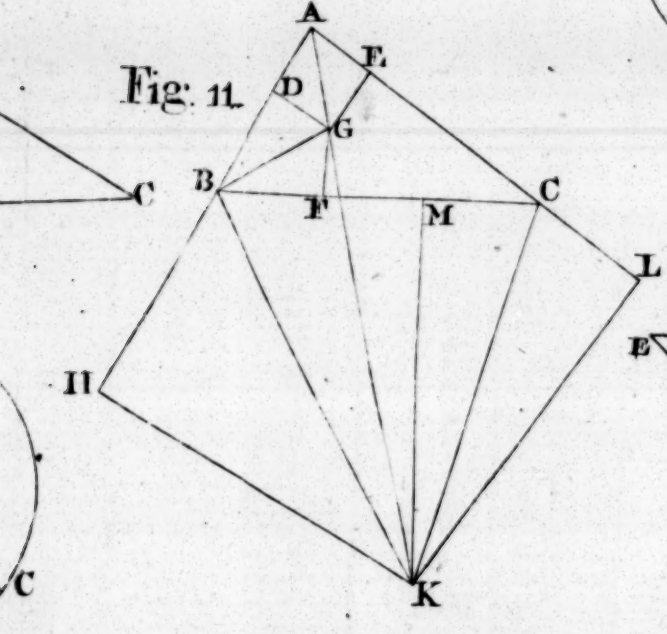

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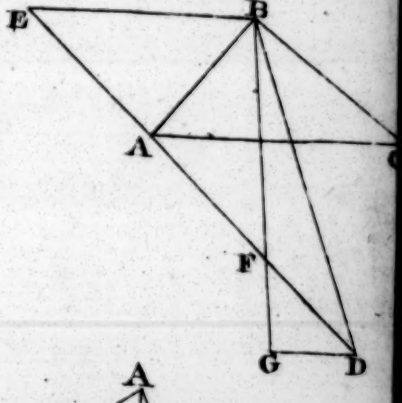

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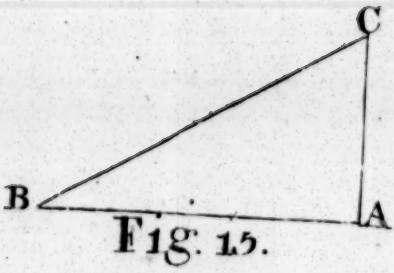

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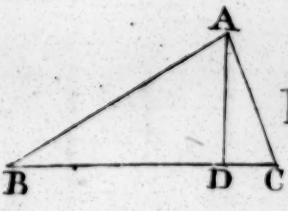

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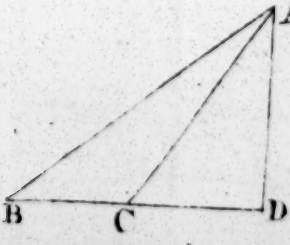

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